International Journal of Applied Research

ISSN Print: 2394-7500
ISSN Online: 2394-5869
Impact Factor: 5.2
IJAR 2016; 2(11): 345-347
www.allresearchjournal.com
Received: 21-09-2016
Accepted: 22-10-2016

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Common fixed point theorems in fuzzy metric spaces using weakly compatible conditions

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Abstract
The aim of this work is to introduce the coincidence and common fixed points of two self maps under weakly compatible condition on fuzzy metric spaces.

Keywords: φ −contractions, coincidence point, common fixed point, fuzzy metric space

1. Introduction
Zadah [8] introduced the concept of fuzzy set. Thereafter, Grabiec [1] defined the G-complete metric space and proved some fixed point results on the fuzzy metric space. Mishra *et al.* [4] also proposed several fixed point theorems for asymptotically commutating maps in the same space. The definition of fuzzy metric space introduced by Kramosil and Michalek [2] also modified. Then, this modified definition given by George and Veeramani is today known as M-Complete fuzzy metric space. Also every G-complete metric space is M-complete fuzzy metric space. George and Veeramani [5] modified the definition of the Cauchy sequence given by Grabiec [1].

Many results of common fixed points have been given in metric spaces, fuzzy metric spaces, partial metric spaces, C*-Algebra-valued b-metric space etc. Banach contraction principle is a classical and celebrated resultof fixed point theory. This result is extended in many spaces i.e. fuzzy metric space, partial metric space, Menger metric space, cone metric space etc. Likewise, in [9] a common fixed point of two mappings such that one mapping is ψ – weak contraction with respect to another mapping has been given.

Firsty, some definitions which are used to prove the main results.

Definition 1.1 [6] A binary operation $T:[0,1]\times[0,1]\rightarrow[0,1]$ is a continuous $t-norm$ if the following conditions hold:
(a) T is commutative and associative,
(b) T is continuous.
(c) T(a, 1) = a for all $a\varepsilon$ [0, 1]
(d) T(a, b)≤ T(a, c)+T(a, d), whenever $a \le c$ and $b \le d$, for $a, b, c, d \in [0,1]$

Definition 1.2 [5] A fuzzy metric space is an ordered triple (X, M, T) such that X is a (nonempty) set, T is a continuous t-norm and M is a fuzzy set on $X\times X\times(0,1)$ satisfying the following conditions, for all x, y, z ∈ X, t > 0:
(FM -1) M (x, y, t) > 0;
(FM -2) M(x, y, t) = 1 if and only if x = y;
(FM -3) M(x, y, t) = M (y, x, t);
(FM -4) M(x, y, t) ∗ M (y, z, s) ≤M(x, z, t + s);
(FM -5) M(x, y, t): $(0,\infty) \rightarrow (0,1)$ is continuous.

Definition 1.3 ([5, 3]) Let (X, M, T) be a fuzzy metric space. Then:
1) A sequence $\{x_n\}$ is said to converge to x in X, denoted by $x_n \rightarrow x$, if and only if $\lim_{n\to\infty} M(x_n, x, t)$ for all t > 0, i.e. for each $r \in (0,1)$ and t > 0, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r$ for all $n \ge n_0$.

2) A sequence $\{x_n\}$ in X is an M-Cauchy sequence if and only if for each $0 < \in < 1$, t > 0, there exists $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1 - \in$ for any m, $n \geq n_0$.
3) The fuzzy metric space (X, M, ∗) is called M-complete if every M-Cauchy sequence is convergent in X.

Definition 1.4 [1] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, T) is G-Cauchy if $\lim_{n\to+\infty}(M_{n+p}, x_n, t)$ for every t > 0 and for every p > 0.

Definition 1.5 [9] Let (X, M, T) be a fuzzy metric space and f, T be two self mappings on X. A point x in X is called a coincidence point (common fixed point) of f and T if fx = Tx (fx = Tx = x). Also the pair of mappings f,T: X→X are said to be weakly compatible if they commute on the set of coincidence points.

Definition 1.6 [7] The function φ: [0, 1] → $[0, 1]$ which is used by altering the distance between two points satisfies the following properties:
(P1) φ is and left continuous and strictly decreasing;
(P2) $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

Obviously, we obtain that $\lim_{\lambda\to 1^-} \varphi(\lambda) = \varphi(1) = 0$.

Theorem 1.7 [7] Let (X, M, T) be an M-complete fuzzy metric space and T a self-map of X and suppose that φ: [0, 1] → $[0, 1]$ satisfies the foregoing properties (P1) and (P2). Furthermore, let k be a function from $(0, \infty)$ into (0, 1). If for any t > 0, T satisfies the following condition:

$$\varphi\big(M(Tx, Ty, t)\big) \leq k(t)\, \varphi(M(x, y, t))$$

where $x, y \in X$ and $x \neq y$, then T has a unique fixed point.

2. The Main Results
In this section, two theorems are proved. The first result establishes coincidence point of two self mappings on fuzzy metric spaces and second result establishes a unique common fixed point of two self mappings on fuzzy metric spaces.

Theorem 2.1 Let (X, M, T) be a fuzzy metric space and suppose φ: [0, 1] → $[0, 1]$ satisfies the foregoing properties (P1) and (P2). Furthermore, let k be a function from (0, ∞) to (0, 1). If T and f are two self maps of X satisfying

$$\varphi\big(M(Tx, Ty, t)\big) \leq k(t)\, \varphi(M(fx, fy, t)) \qquad (2.1)$$

where $x, y \in X$ and $x \neq y$ and f(X) is a G-complete subspace of X, then f and T have coincidence point in X. Also this fixed point is unique.

Proof Let x_0 is an arbitrary point in X. Let $x_1 \in X$ such that $Tx_0 = fx_1$. This is possible f(X)⊆T(X). Continuing, this process indefinitely, for every x_n in X, one can find a x_{n+1} such that $y_n = Tx_n = fx_{n+1}$. Without loss of generality, one may assume that $y_{n+1} \neq y_n$ for all $n \in N$, otherwise f and T have a coincidence point. In case, $y_{n+1} \neq y_n$. Now from (2.1), we have

$$\varphi\big(M(y_n, y_{n+1}, t)\big) = k(t)\, \varphi\big(M(Tx_n, Ty_n, t)\big) \leq k(t)\, \varphi\big(M(fx_n, fx_{n+1}, t)\big) < \varphi(M(y_{n-1}, y_n, t)).$$

This implies
$$\varphi\big(M(y_n, y_{n+1}, t)\big) < \varphi(M(y_{n-1}, y_n, t)). \qquad (2.2)$$

But, φ is strictly decreasing function. This implies that
$\varphi\big(M(y_n, y_{n+1}, t)\big) > \varphi(M(y_{n-1}, y_n, t))$ for all $n \in N$.

This implies that $\{M(y_n, y_{n+1}, t)\}$ is increasing sequence of positive real numbers in [0, 1]. Let $S(t) = \lim_{n\to\infty} M(y_n, y_{n+1}, t)$. Now, we show that S(t) = 1 for all t > 0. Otherwise, there must exist some t > 0 such that S(t) < 1. Taking n → ∞ in (2.2), we obtain $\varphi(S(t)) < \varphi(S(t))$, a contradiction.

Therefore $M(y_n, y_{n+1}, t) = 1$ as n → ∞.

Note that, for each positive integer p,
$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) \ldots M(y_{n+p-1}, y_{n+p}, t/p).$$

This implies that
$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * \ldots * 1 = 1.$$

Therefore $\{y_n\}$ is a G-Cauchy sequence. Since f(X) is G-complete, therefore there exists $q \rightarrow f(X)$ such that $y_n \rightarrow q$ as n → ∞. Consequently, we obtain p in X such that fp = q. Next we show that p is coincidence point of f and T. Now using (2.1), we have

$$\varphi\big(M(Tp,\ fx_{n+1},t)\big) = \mathrm{k(t)}\ \varphi\big(M(Tp,Tx_n,t)\big) \leq \mathrm{k(t)}\ \varphi(M(fp,fx_n,t)).$$

This implies,

$$\lim_{n\rightarrow\infty} \varphi\big(M(Tp,\ fx_{n+1},t)\big)\ \leq\ \lim_{n\rightarrow\infty} \mathrm{k(t)}\ \varphi(M(fp,fx_n,t)).$$

This implies,

$0 \leq\ \varphi\big(M(Tp,fp,t)\big) \leq \mathrm{k(t)}\ \varphi(M(fp,fp,t)) = \mathrm{k(t)}\ \varphi(1) = 0.$

So $\varphi\big(M(Tp,fp,t)\big) = 0.$

So by property (P2) of φ, we have

M(Tp, fp, t) = 1

This implies Tp = fp. This completes the proof the theorem.

Now we obtain a point p in X such that Tp = fp = q (say) which further implies fTp = Tfp, since f and T are weakly compatible. Obviously Tq = fq.

Now we show that fq = q, If not, then

$$\varphi\big(M(fq,q,t)\big) = \mathrm{k(t)}\ \varphi\big(M(Tq,Tp,t)\big)\ \leq \mathrm{k(t)}\ \varphi(M(fq,fp,t)) = \mathrm{k(t)}\ \varphi\big(M(fq,q,t)\big). \quad (2.4)$$

This implies that

$\varphi(M(fq,q,t) =\ 0.$

So by properties (P1) of φ, we obtain

M (fq, q, t) = 1.

This implies fq = q.

Now we prove uniqueness of the theorem. Let y be another common fixed point of f and T i.e. fy = Ty = y.

Now

$\varphi\big(M(y,q,t)\big) = \mathrm{k(t)}\ \varphi\big(M(Ty,Tp,t)\big) \leq \mathrm{k(t)}\ \varphi(M(fy,fp,t))$

$<\ \varphi\big(M(y,q,t)\big)$ a contradiction which proves y = q. This completes the proof of the theorem.

Example 2.2 Let X = [0, 1] and a ∗ b = min{a, b}. Let M be the standard fuzzy metric space induced by d where d(x, y) = |x – y| for x, y ∈ X. Then (X, M, T) is the complete fuzzy metric space. Let f (t) = $1-\sqrt{t}$ for all t ∈ [0, 1].

Also, let

fx = $\frac{1}{2}$ (1 - x), $x\ \in [0,1]$ and Tx = $\frac{1}{3}$, for all $x\ \in [0,1]$

Then f and T satisfy all the conditions of the theorem 2.2. Note that 1/3 is the coincidence point which also turns out to be common fixed point also.

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