International Journal of Applied Research 2016; 2(11): 462-466

ISSN Print: 2394-7500
ISSN Online: 2394-5869
Impact Factor: 5.2
IJAR 2016; 2(11): 462-466
www.allresearchjournal.com
Received: 08-09-2016
Accepted: 09-10-2016

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On the negative Pell equation $y^2 = 21x^2 - 3$

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Abstract

The binary quadratic equation represented by the negative Pellian $y^2 = 21x^2 - 3$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas.

Keywords: Binary quadratic, Pell equation, integer solutions, Hyperbola, Parabola

1. Introduction

Pell's equation is any Diophantine equation of the form $x^2 - ny^2 = 1$, when n is a given positive non-square integer has always positive integer solutions. This equation was first studied extensively in India, starting with Brahmagupta, who developed the Chakravala method to Pell's equation and other quadratic indeterminate equations. When k is a positive integer and $n \in (k^2 \pm 4, k^2 \pm 1)$, positive integer solutions of the equations $x^2 - ny^2 = \pm 4$ and $x^2 - ny^2 = \pm 1$, have been investigated by Jones in [2]. In [1, 4, 6-8] some special Pell equation and their solutions are considered. J.L. Lagrange proved that the positive Pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct solutions whereas the negative Pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [3], an elementary proof of a ceriterium for the solvability of the Pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For example, the equations $y^2 = 3x^2 - 1$, $y^2 = 7x^2 - 4$ have no integer solutions whereas $y^2 = 65x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [5, 9-11]. More specifically, one may refer, "The On-line Encyclopedia of integer sequences" (A031396,A130226,A031398) for values of D for which the negative Pell equation $y^2 = Dx^2 - 1$ is solvable or not.

In this communication, the negative Pell equation given by $y^2 = 21x^2 - 3$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

2. Method of Analysis

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 21x^2 - 3 \qquad (1)$$

whose smallest positive integer solution is $x_0 = 2, \quad y_0 = 9$.

To obtain the other solutions of (1), consider the Pellian equation

$$y^2 = 21x^2 + 1$$

whose initial solution is $\tilde{x}_0 = 12, \quad \tilde{y}_0 = 55$ and the general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{21}} g_n, \quad \tilde{y}_n = \frac{1}{2} f_n$$

where $f_n = \left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right], \quad g_n = \left[\left(55+12\sqrt{21}\right)^{n+1} - \left(55-12\sqrt{21}\right)^{n+1}\right]$

Applying Brahamagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$y_{n+1} = \frac{1}{2}\left[9 f_n + 2\sqrt{21} g_n\right]$$

$$x_{n+1} = \frac{1}{2\sqrt{21}}\left[2\sqrt{21} f_n + 9 g_n\right]$$

Some numerical examples of x and y satisfying (1) are given in the table below

n	x_{n+1}	y_{n+1}
0	218	999
1	23978	109881
2	2637362	12085911

3. Observations

We observe some interesting relations among the solutions which are presented below

1. The recurrence relations satisfied by the solutions of (1) are given by

$$x_{n+3} - 110x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 110y_{n+2} + y_{n+1} = 0$$

2. $24y_{n+2} = x_{n+3} - x_{n+1}$.
3. $504x_{n+2} = y_{n+3} - y_{n+1}$.
4. Each of the following expressions is a perfect square.

i) $\frac{84x_{2n+2} - 18y_{2n+2} + 6}{3}$

ii) $\frac{4662x_{2n+2} - 42x_{2n+3} + 168}{84}$

iii) $\frac{512778x_{2n+2} - 42x_{2n+4} + 18480}{9240}$

iv) $\frac{4615002x_{2n+3} - 41958x_{2n+4} + 1512}{756}$

v) $\frac{3052x_{2n+2} - 6y_{2n+3} + 110}{55}$

vi) $\frac{335692x_{2n+2} - 6y_{2n+4} + 12098}{6049}$

vii) $\frac{28x_{2n+3} - 666y_{2n+2} + 110}{55}$

viii) $3052x_{2n+3} - 666y_{2n+3} + 2$

ix) $\frac{335692x_{2n+3} - 666y_{2n+4} + 110}{55}$

x) $\frac{28x_{2n+4} - 73254y_{2n+2} + 12098}{6049}$

xi) $\frac{3052x_{2n+4} - 73254y_{2n+3} + 110}{55}$

xii) $335692x_{2n+4} - 73254y_{2n+4} + 2$

xiii) $\frac{y_{2n+3} - 109y_{2n+2} + 18}{9}$

xiv) $\frac{y_{2n+4}-11989y_{2n+2}+1980}{990}$

xv) $\frac{109y_{2n+4}-11989y_{2n+3}+18}{9}$

5. Each of the following expressions is a cubical integer.

i) $28x_{3n+3}-6y_{3n+3}+330$

ii) $\frac{111x_{3n+3}-x_{3n+4}+660}{2}$

iii) $\frac{12209x_{3n+3}-x_{3n+5}+72600}{220}$

iv) $\frac{12209x_{3n+4}-111x_{3n+5}+660}{2}$

v) $\frac{9156x_{3n+3}-18y_{3n+4}+54450}{165}$

vi) $\frac{1007076x_{3n+3}-18y_{3n+5}+598851}{18147}$

vii) $\frac{84x_{3n+4}-1998y_{3n+3}+54450}{165}$

viii) $\frac{9156x_{3n+4}-1998y_{3n+4}+990}{3}$

ix) $\frac{1007076x_{3n+4}-1998y_{3n+5}+54450}{165}$

x) $\frac{84x_{3n+5}-219762y_{3n+3}+5988510}{18147}$

xi) $\frac{9156x_{3n+5}-219762y_{3n+4}+54450}{165}$

xii) $\frac{1007076x_{3n+5}-219762y_{3n+5}+990}{3}$

xiii) $\frac{2y_{3n+5}-23978y_{3n+3}+653400}{1980}$

xiv) $\frac{2y_{3n+4}-218y_{3n+3}+5940}{18}$

xv) $\frac{436y_{3n+5}-47956y_{3n+4}+11880}{36}$

4. Remarkable Observations

On employing linear combinations among the solutions of (1), one may generate integer solutions for our choices of hyperbolas which are presented in the table I below:

Table I

Hyperbola	(X,Y)
$X^2-Y^2=36$	i) $\left(84x_{n+1}-18y_{n+1},\ \frac{84y_{n+1}-378x_{n+1}}{\sqrt{21}}\right)$ ii) $\left(9156x_{n+2}-1998y_{n+2},\ \frac{9156y_{n+2}-41958x_{n+2}}{\sqrt{21}}\right)$ iii) $\left(1007076x_{n+3}-219762y_{n+3},\ \frac{1007076y_{n+3}-4615002x_{n+3}}{\sqrt{21}}\right)$

$27X^2 - 588Y^2 = 762048$	$\left(4662x_{n+1} - 42x_{n+2}, \quad \frac{42x_{n+2} - 4578x_{n+1}}{\sqrt{21}}\right)$
$X^2 - Y^2 = 63504$	$\left(769167x_{n+2} - 6993x_{n+3}, \quad \frac{32046x_{n+3} - 3524766x_{n+2}}{\sqrt{21}}\right)$
$X^2 - Y^2 = 108900$	i) $\left(9156x_{n+1} - 18y_{n+2}, \quad \frac{84y_{n+2} - 41958x_{n+1}}{\sqrt{21}}\right)$ ii) $\left(84x_{n+2} - 1998y_{n+1}, \quad \frac{9156y_{n+1} - 378x_{n+2}}{\sqrt{21}}\right)$ iii) $\left(1007076x_{n+2} - 1998y_{n+3}, \quad \frac{9156y_{n+3} - 4615002x_{n+2}}{\sqrt{21}}\right)$ iv) $\left(9156x_{n+3} - 219762y_{n+2}, \quad \frac{1007076y_{n+2} - 41958x_{n+3}}{\sqrt{21}}\right)$
$X^2 - Y^2 = 1317254436$	i) $\left(1007076x_{n+1} - 18y_{n+3}, \quad \frac{84y_{n+3} - 4615002x_{n+1}}{\sqrt{21}}\right)$ ii) $\left(84x_{n+3} - 219762y_{n+1}, \quad \frac{1007076y_{n+1} - 378x_{n+3}}{\sqrt{21}}\right)$
$4X^2 - 81Y^2 = 5184$	$\left(2y_{n+2} - 218y_{n+1}, \quad \frac{222y_{n+1} - 2y_{n+2}}{\sqrt{21}}\right)$
$484X^2 - 9801Y^2 = 1897473600$	$\left(y_{n+3} - 11989y_{n+1}, \quad \frac{12209y_{n+1} - y_{n+3}}{\sqrt{21}}\right)$
$X^2 - Y^2 = 5184$	$\left(436y_{n+3} - 47956y_{n+2}, \quad \frac{219762y_{n+2} - 1998y_{n+3}}{\sqrt{21}}\right)$

On employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the table II below:

Table II

Parabola	(X, Y)
$Y^2 = 3X - 36$	$\left(84x_{2n+2} - 18y_{2n+2} + 6, \quad \frac{84y_{n+1} - 378x_{n+1}}{\sqrt{21}}\right)$
$7Y^2 = 27X - 9072$	$\left(4662x_{2n+2} - 42x_{2n+3} + 168, \quad \frac{42x_{n+2} - 4578x_{n+1}}{\sqrt{21}}\right)$
$Y^2 = 126X - 63504$	$\left(769167x_{2n+3} - 6993x_{2n+4} + 252, \quad \frac{32046x_{n+3} - 3524766x_{n+2}}{\sqrt{21}}\right)$
$Y^2 = 55X - 12100$	i) $\left(3052x_{2n+2} - 6y_{2n+3} + 110, \quad \frac{28y_{n+2} - 13986x_{n+1}}{\sqrt{21}}\right)$ ii) $\left(28x_{2n+3} - 666y_{2n+2} + 110, \quad \frac{3052y_{n+1} - 126x_{n+2}}{\sqrt{21}}\right)$ iii) $\left(335692x_{2n+3} - 666y_{2n+4} + 110, \quad \frac{3052y_{n+3} - 1538334x_{n+2}}{\sqrt{21}}\right)$ iv) $\left(3052x_{2n+4} - 73254y_{2n+3} + 110, \quad \frac{335692y_{n+2} - 13986x_{n+3}}{\sqrt{21}}\right)$

$Y^2 = 6049X - 146361604$	i) $\left(335692\mathrm{x}_{2n+2} - 6y_{2n+4} + 12098,\ \frac{28y_{n+3} - 1538334x_{n+1}}{\sqrt{21}}\right)$ ii) $\left(28\mathrm{x}_{2n+4} - 73254y_{2n+2} + 12098,\ \frac{335692y_{n+1} - 126x_{n+3}}{\sqrt{21}}\right)$
$Y^2 = X - 4$	i) $\left(3052\mathrm{x}_{2n+3} - 666y_{2n+3} + 2,\ \frac{3052y_{n+2} - 13986x_{n+2}}{\sqrt{21}}\right)$ ii) $\left(335692\mathrm{x}_{2n+4} - 73254y_{2n+4} + 2, \frac{335692y_{n+3} - 1538334x_{n+3}}{\sqrt{21}}\right)$
$9Y^2 = 16X - 576$	$\left(y_{2n+3} - 109y_{2n+2} + 18,\ \frac{222y_{n+1} - 2y_{n+2}}{\sqrt{21}}\right)$
$9Y^2 = 1760X - 6969600$	$\left(y_{2n+4} - 11989y_{2n+2} + 1980,\ \frac{24418y_{n+1} - 2y_{n+3}}{\sqrt{21}}\right)$
$Y^2 = 16X - 576$	$\left(109y_{2n+4} - 11989y_{2n+3} + 18,\ \frac{73254y_{n+2} - 666y_{n+3}}{\sqrt{21}}\right)$

5. Conclusion
In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation $y^2 = 21x^2 - 3$. As the binary quadratic diophantine equations are rich in variety, one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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