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General solution and generalized Hyers-Ulam stability of new n-type additive quadratic functional equation stable in a Banach space: Using direct method

V Govindan, S Murthy, M Arunkumar

Abstract
In this paper, the authors introduce and investigate the general solution and generalized Hyers-Ulam stability of a generalized new *n*-type additive-quadratic functional equation of the form

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)$$
$$+f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=\frac{n}{n-1}[f(x)-f(-x)]+\frac{n+1}{n}[f(y)-f(-y)]+\frac{n+2}{n+1}[f(z)-f(-z)]$$
$$+2\left(\frac{n}{n-1}\right)^2[f(x)+f(-x)]+2\left(\frac{n+1}{n}\right)^2[f(y)+f(-y)]+2\left(\frac{n+2}{n+1}\right)^2[f(z)+f(-z)]$$

where n is a positive integer with $n \neq 0$, in Banach Space using direct methods.
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1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [33] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [18, 19]. The result of Hyers was generalized by Aoki [2] for approximate additive mappings and Rassias [28, 29] for approximate linear mapping by allowing the Cauchy difference operator $CDf(x,y)=f(x+y)-[f(x)+f(y)]$ to be controlled by $\in\left(\|x\|^p+\|y\|^p\right)$.

The functional equation

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

is related to additive function [15]. It is a natural that such question is called a additve functional equation. In particular, every solution of the additive equation (1.1) is said to be additive function

In 1994, a further generalization was obtained by Gavruta [5], who replaced $\in\left(\|x\|^p+\|y\|^p\right)$ by a general control function $\phi(x,y)$. Rassias [28-30] treated the Ulam-Gavruta-Rassias stability on linear and non-linear mappings and generalized Hyers result. The reader is referred to the following books and research articles which provide an existence account of progress made on Ulam's problem during the last seventy years (cf) [8-33].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.2}$$

is related to symmetric biadditive function [12]. It is a natural that such question is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be quadratic function.

It is well known that a function f between real vector spaces in quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$ for all x (see) [10]. The biadditive function B is given by $B(x,x)=\frac{1}{4}[f(x+y)+f(x-y)]$. In [14, 15], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation (1.2). A Hyers-Ulam stability problem for the quadratic functional equation (1.2) was proved by Skof for function $f:E_1\to E_2$, where E_1 is a normed space and E_2 a Banach space (see) [21-23]. P.W. Cholewa [13] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an abelian group. The quadratic functional equation and several other functional equations are useful to characteristic inner product space (cf) [8, 24, 28-32].

In this paper the authors, we introduce and investigate that the general solution and generalized Ulam-Hyers stability of a generalized additive Quadratic functional equation of the form

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)$$
$$+f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=\frac{n}{n-1}[f(x)-f(-x)]+\frac{n+1}{n}[f(y)-f(-y)]+\frac{n+2}{n+1}[f(z)-f(-z)]$$
$$+2\left(\frac{n}{n-1}\right)^2[f(x)+f(-x)]+2\left(\frac{n+1}{n}\right)^2[f(y)+f(-y)]+2\left(\frac{n+2}{n+1}\right)^2[f(z)+f(-z)] \quad (1.3)$$

where n is a positive integer, in Banach Space with direct method.

2. General Solution of the Functional Equation (1.3)

Theorem 2.1: Let X and Y be the real vector space, the additive mapping $f:X\to Y$ satisfies the functional equation (1.1) for all $x,y\in X$, if and only if $f:X\to Y$ satisfies the functional (1.3) equation, for all $x,y,z\in X$ with $f(0)=0$.

Proof. Let $f:X\to Y$ satisfies the functional equation (1.1). Setting (x,y) by $(0,0)$ in (1.1) we obtain $f(0)=0$. Set (x,y) by $(x,-x)$ in (1.1), we get $f(-x)=-f(x)$ for all $x\in X$. Therefore f is an odd function. Replacing (x,y) by (x,x) and $(x,2x)$ in (1.1) we have,

$$f(2x)=2f(x)\ ;\ f(3x)=3f(x) \quad (2.1)$$

for all $x\in X$. In general for any positive integer a, we have

$$f(ax)=af(x) \quad (2.2)$$

for all $x\in X$. Replacing (x,y) by $\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y,\frac{(n+2)}{n+1}z\right)$ in (1.1), we get

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y\right)+f\left(\frac{(n+2)}{n+1}z\right) \quad (2.3)$$

for all $x,y,z\in X$. But we know that from the equation (2.3), we have,

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y\right)=f\left(\frac{n}{n-1}x\right)+f\left(\frac{(n+1)}{n}y\right) \quad (2.4)$$

for all $x,y\in X$. Substituting the value of (2.3) in (2.4), we get

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=f\left(\frac{n}{n-1}x\right)+f\left(\frac{(n+1)}{n}y\right)+f\left(\frac{(n+2)}{n+1}z\right)$$
$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=\frac{n}{n-1}f(x)+\frac{(n+1)}{n}f(y)+\frac{(n+2)}{n+1}f(z) \quad (2.5)$$

for all $x,y,z\in X$. Again replacing (x,y) by $\left(\frac{n}{n-1}x,\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)$ in equation (1.1), we get,

$$f\left(\frac{(n+1)}{n}y+\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)=f\left(\frac{(n+1)}{n}y\right)+f\left(\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right) \quad (2.6)$$

for all $x,y,z\in X$. But we know that from the equation (2.5), we have

$$f\left(\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)=f\left(\frac{n}{n-1}x\right)+f\left(-\frac{(n+2)}{n+1}z\right) \quad (2.7)$$

for all $x,y,z\in X$. If f is an odd function we have

$$f\left(\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)=f\left(\frac{n}{n-1}x\right)-f\left(\frac{(n+2)}{n+1}z\right) \quad (2.8)$$

for all $x,z\in X$. Substituting the value of (2.6) in (2.8), we get,

$$f\left(\frac{(n+1)}{n}y+\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)=f\left(\frac{(n+1)}{n}y\right)+f\left(\frac{n}{n-1}x\right)-f\left(\frac{(n+2)}{n+1}z\right)$$

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)=f\left(\frac{n}{n-1}x\right)+f\left(\frac{(n+1)}{n}y\right)-f\left(\frac{(n+2)}{n+1}z\right)$$

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)=\frac{n}{n-1}f(x)+\frac{(n+1)}{n}f(y)-\frac{(n+2)}{n+1}f(z) \quad (2.9)$$

for all $x, y, z \in X$. Replacing (x, y) by $\left(\frac{(n+2)}{n+1}z, \frac{n}{n-1}x-\frac{(n+1)}{n}y\right)$ in (1.1), we get

$$f\left(\frac{(n+2)}{n+1}z+\frac{n}{n-1}x-\frac{(n+1)}{n}y\right)=f\left(\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y\right) \quad (2.10)$$

for all $x, y, z \in X$. But we know that from the equation (2.10), we arrive

$$f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y\right)=f\left(\frac{n}{n-1}x\right)+f\left(-\frac{(n+1)}{n}y\right) \quad (2.11)$$

for all $x, y, z \in X$. If f is an odd function, then

$$f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y\right)=f\left(\frac{n}{n-1}x\right)-f\left(\frac{(n+1)}{n}y\right) \quad (2.12)$$

for all $x, y \in X$. Substituting the value of (2.10) in (2.12), we get

$$f\left(\frac{(n+2)}{n+1}z+\frac{n}{n-1}x-\frac{(n+1)}{n}y\right)=f\left(\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x\right)-f\left(\frac{(n+1)}{n}y\right)$$

$$f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=f\left(\frac{n}{n-1}x\right)-f\left(\frac{(n+1)}{n}y\right)+f\left(\frac{(n+2)}{n+1}z\right)$$

$$f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=\frac{n}{n-1}f(x)-\frac{(n+1)}{n}f(y)+\frac{(n+2)}{n+1}f(z) \quad (2.13)$$

for all $x, y, z \in X$. Setting (x, y) by $\left(\frac{(n+2)}{n+1}z, \frac{(n+1)}{n}y-\frac{n}{n-1}x\right)$ in (1.1), we get

$$f\left(\frac{(n+2)}{n+1}z+\frac{(n+1)}{n}y-\frac{n}{n-1}x\right)=f\left(\frac{(n+2)}{n+1}z\right)+f\left(\frac{(n+1)}{n}y-\frac{n}{n-1}x\right) \quad (2.14)$$

for all $x, y, z \in X$. But we know that from the equation (2.6), we have

$$f\left(\frac{(n+1)}{n}y-\frac{n}{n-1}x\right)=f\left(\frac{(n+1)}{n}y\right)+f\left(-\frac{n}{n-1}x\right) \quad (2.15)$$

for all $x, y, z \in X$. If f is an odd function, that

$$f\left(\frac{(n+1)}{n}y-\frac{n}{n-1}x\right)=f\left(\frac{(n+1)}{n}y\right)-f\left(\frac{n}{n-1}x\right) \quad (2.16)$$

for all $x, y \in X$. Substituting the value of (2.14) in (2.16), we get

$$f\left(\frac{(n+2)}{n+1}z+\frac{(n+1)}{n}y-\frac{n}{n-1}x\right)=f\left(\frac{(n+2)}{n+1}z\right)+f\left(\frac{(n+1)}{n}y\right)-f\left(\frac{n}{n-1}x\right)$$

$$f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=-f\left(\frac{n}{n-1}x\right)+f\left(\frac{(n+1)}{n}y\right)+f\left(\frac{(n+2)}{n+1}z\right)$$

$$f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=-\frac{n}{n-1}f(x)+\frac{(n+1)}{n}f(y)+\frac{(n+2)}{n+1}f(z) \quad (2.17)$$

for all $x, y, z \in X$. Adding the equations (2.17), (2.13) and (2.9), (2.5), using the oddness and remodifying, we arrive desired our result.

Conversely, $f : X \to Y$ satisfies the functional equation (1.1) with $f(0) = 0$. For assume that $f : X \to Y$ satisfies the functional equation (3.1) with $f(0) = 0$; Replacing (x, y, z) by $(x, 0, 0)$, $(0, x, 0)$ and $(0, 0, x)$, respectively in (3.0), we obtain

$$f\left(\frac{n}{n-1}x\right)=\frac{n}{n-1}f(x);\ f\left(\frac{n+1}{n}y\right)=\frac{n+1}{n}f(y) \text{ and } f\left(\frac{n+2}{n+1}z\right)=\frac{n+2}{n+1}f(z) \quad (2.18)$$

for all $x, y, z \in X$. Replacing (x, y, z) by $\left(\frac{(n-1)x}{n}, \frac{nx}{n+1}, 0\right)$ in (1.3), we arrive (1.1).

Theorem: Let X and Y be the real vector space, the mapping $f : X \to Y$ satisfies the functional equation (1.2) *for all* $x, y \in X$ *, if and only if* $f : X \to Y$ *satisfies the functional equation(1.3), for all* $x, y, z \in X$ *with* $f(0) = 0$.

Proof. Let $f : X \to Y$ satisfies the functional equation (1.2). Setting (x, y) by $(0, 0)$ in (1.2) we obtain $f(0) = 0$. Setting (x, y) by $(0, y)$ in (1.2), we get $f(-y) = f(y)$ for all $y \in X$. Therefore f is an even function. Replacing (x, y) by (x, x) and $(x, 2x)$ in (3.4) we obtain,

$$f(2x) = 4f(x)\ ;\ f(3x) = 9f(x) \quad (2.19)$$

for all $x \in X$. In general for any positive integer a , such that

$$f(ax) = a^2 f(x) \tag{2.20}$$

for all $x \in X$. Replacing (x, y) by $\left(\frac{n}{n-1}x, \frac{(n+1)}{n}y \right)$ in (1.2), we obtain

$$f\left(\frac{n}{n-1}x + \frac{(n+1)}{n}y \right) + f\left(\frac{n}{n-1}x - \frac{(n+1)}{n}y \right) = 2f\left(\frac{n}{n-1}x \right) + 2f\left(\frac{(n+1)}{n}y \right)$$

$$f\left(\frac{n}{n-1}x + \frac{(n+1)}{n}y \right) + f\left(\frac{n}{n-1}x - \frac{(n+1)}{n}y \right) = 2\left(\frac{n}{n-1} \right)^2 f(x) + 2\left(\frac{(n+1)}{n} \right)^2 f(y) \tag{2.21}$$

for all $x, y \in X$. Setting (x, y) by $\left(\frac{n}{n-1}x, \frac{(n+2)}{n+1}z \right)$ in (1.2), we get

$$f\left(\frac{n}{n-1}x + \frac{(n+2)}{n+1}z \right) + f\left(\frac{n}{n-1}x - \frac{(n+2)}{n+1}z \right) = 2f\left(\frac{n}{n-1}x \right) + 2f\left(\frac{(n+2)}{n+1}z \right)$$

$$f\left(\frac{n}{n-1}x + \frac{(n+2)}{n+1}z \right) + f\left(\frac{n}{n-1}x - \frac{(n+2)}{n+1}z \right) = 2\left(\frac{n}{n-1} \right)^2 f(x) + 2\left(\frac{(n+2)}{n+1} \right)^2 f(z) \tag{2.22}$$

for all $x, z \in X$. Replacing (x, y) by $\left(\frac{(n+1)}{n}y, \frac{(n+2)}{n+1}z \right)$ in (1.2), we arrive

$$f\left(\frac{(n+1)}{n}y + \frac{(n+2)}{n+1}z \right) + f\left(\frac{(n+1)}{n}y - \frac{(n+2)}{n+1}z \right) = 2f\left(\frac{(n+1)}{n}y \right) + 2f\left(\frac{(n+2)}{n+1}z \right) \tag{2.23}$$

for all $y, z \in X$. From the equation (2.23), that

$$f\left(\frac{(n+1)}{n}y + \frac{(n+2)}{n+1}z \right) + f\left(\frac{(n+1)}{n}y - \frac{(n+2)}{n+1}z \right) = 2\left(\frac{(n+1)}{n} \right)^2 f(y) + 2\left(\frac{(n+2)}{n+1} \right)^2 f(z) \tag{2.24}$$

for all $y, z \in X$. Adding the equations (2.24), (2.23) and (2.22), and remodifying we arrive

$$f\left(\frac{n}{n-1}x + \frac{(n+1)}{n}y \right) + f\left(\frac{n}{n-1}x - \frac{(n+1)}{n}y \right) + f\left(\frac{n}{n-1}x + \frac{(n+2)}{n+1}z \right) + f\left(\frac{n}{n-1}x - \frac{(n+2)}{n+1}z \right)$$

$$+ f\left(\frac{(n+1)}{n}y + \frac{(n+2)}{n+1}z \right) + f\left(\frac{(n+1)}{n}y - \frac{(n+2)}{n+1}z \right) = 2\left(\frac{n}{n-1} \right)^2 f(x) + 2\left(\frac{(n+1)}{n} \right)^2 f(y)$$

$$+2\left(\frac{n}{n-1} \right)^2 f(x) + 2\left(\frac{(n+2)}{n+1} \right)^2 f(z) + 2\left(\frac{(n+1)}{n} \right)^2 f(y) + 2\left(\frac{(n+2)}{n+1} \right)^2 f(z)$$

$$f\left(\frac{n}{n-1}x + \frac{(n+1)}{n}y \right) + f\left(\frac{n}{n-1}x - \frac{(n+1)}{n}y \right) + f\left(\frac{n}{n-1}x + \frac{(n+2)}{n+1}z \right) + f\left(\frac{n}{n-1}x - \frac{(n+2)}{n+1}z \right)$$

$$+ f\left(\frac{(n+1)}{n}y + \frac{(n+2)}{n+1}z \right) + f\left(\frac{(n+1)}{n}y - \frac{(n+2)}{n+1}z \right) = 4\left(\frac{n}{n-1} \right)^2 f(x) + 4\left(\frac{(n+1)}{n} \right)^2 f(y) + 4\left(\frac{(n+2)}{n+1} \right)^2 f(z) \tag{2.25}$$

for all $x, y, z \in X$. Adding $f\left(\frac{(n+2)}{n+1}z \right)$ in equation (2.25) on both sides, we get

$$f\left(\frac{n}{n-1}x + \frac{(n+1)}{n}y \right) + f\left(\frac{(n+2)}{n+1}z \right) + f\left(\frac{n}{n-1}x - \frac{(n+1)}{n}y \right) + f\left(\frac{n}{n-1}x + \frac{(n+2)}{n+1}z \right)$$

$$+ f\left(\frac{n}{n-1}x - \frac{(n+2)}{n+1}z \right) + f\left(\frac{(n+1)}{n}y + \frac{(n+2)}{n+1}z \right) + f\left(\frac{(n+1)}{n}y - \frac{(n+2)}{n+1}z \right)$$

$$= 4\left(\frac{n}{n-1} \right)^2 f(x) + 4\left(\frac{(n+1)}{n} \right)^2 f(y) + 4\left(\frac{(n+2)}{n+1} \right)^2 f(z) + f\left(\frac{(n+2)}{n+1}z \right) \tag{2.26}$$

for all $x, y, z \in X$. Multiply by 2 in (2.26) on both sides, we obtain

$$2f\left(\frac{n}{n-1}x + \frac{(n+1)}{n}y \right) + 2f\left(\frac{(n+2)}{n+1}z \right) + 2f\left(\frac{n}{n-1}x - \frac{(n+1)}{n}y \right) + 2f\left(\frac{n}{n-1}x + \frac{(n+2)}{n+1}z \right)$$

$$+2f\left(\frac{n}{n-1}x - \frac{(n+2)}{n+1}z \right) + 2f\left(\frac{(n+1)}{n}y + \frac{(n+2)}{n+1}z \right) + 2f\left(\frac{(n+1)}{n}y - \frac{(n+2)}{n+1}z \right)$$

$$= 8\left(\frac{n}{n-1} \right)^2 f(x) + 8\left(\frac{(n+1)}{n} \right)^2 f(y) + 8\left(\frac{(n+2)}{n+1} \right)^2 f(z) + 2f\left(\frac{(n+2)}{n+1}z \right) \tag{2.27}$$

for all $x, y, z \in X$. But we know that from the equation (2.27), we arrive

$$f\left(\frac{n}{n-1}x + \frac{(n+1)}{n}y + \frac{(n+2)}{n+1}z \right) + f\left(\frac{n}{n-1}x + \frac{(n+1)}{n}y - \frac{(n+2)}{n+1}z \right) = 2f\left(\frac{n}{n-1}x + \frac{(n+1)}{n}y \right) + 2f\left(\frac{(n+2)}{n+1}z \right) \tag{2.28}$$

for all $x, y, z \in X$. Substituting the value of (2.27) in (2.26), that

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y\right)$$
$$+2f\left(\frac{n}{n-1}x+\frac{(n+2)}{n+1}z\right)+2f\left(\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)$$
$$+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)==8\left(\frac{n}{n-1}\right)^2 f(x)+8\left(\frac{(n+1)}{n}\right)^2 f(y)+8\left(\frac{(n+2)}{n+1}\right)^2 f(z)+2\left(\frac{(n+2)}{n+1}\right)^2 f(z) \quad (2.29)$$

for all $x, y, z \in X$. Adding $2f\left(\frac{(n+2)}{n+1}z\right)$ in equation (2.29) on both sides, we get

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+2)}{n+1}z\right)+2f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y\right)$$
$$+2f\left(\frac{n}{n-1}x+\frac{(n+2)}{n+1}z\right)+2f\left(\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)$$
$$=8\left(\frac{n}{n-1}\right)^2 f(x)+8\left(\frac{(n+1)}{n}\right)^2 f(y)+10\left(\frac{(n+2)}{n+1}\right)^2 f(z)+2\left(\frac{(n+2)}{n+1}\right)^2 f(z) \quad (2.30)$$

for all $x, y, z \in X$. But we know that from the equation (2.30), we have

$$f\left(\frac{(n+2)}{n+1}z+\frac{n}{n-1}x+\frac{(n+1)}{n}y\right)+f\left(\frac{(n+2)}{n+1}z-\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y\right)\right)=2f\left(\frac{(n+2)}{n+1}z\right)$$
$$+2f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y\right) \quad (2.31)$$

for all $x, y, z \in X$. Substituting the value of (2.31) in (2.30), we obtain

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{(n+2)}{n+1}z+\frac{n}{n-1}x+\frac{(n+1)}{n}y\right)$$
$$+f\left(\frac{(n+2)}{n+1}z-\frac{n}{n-1}x-\frac{(n+1)}{n}y\right)+2f\left(\frac{n}{n-1}x+\frac{(n+2)}{n+1}z\right)+2f\left(\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right) \quad (2.32)$$
$$+2f\left(\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)=8\left(\frac{n}{n-1}\right)^2 f(x)$$
$$+8\left(\frac{(n+1)}{n}\right)^2 f(y)+12\left(\frac{(n+2)}{n+1}\right)^2 f(z)$$

for all $x, y, z \in X$. Adding $2f\left(\frac{(n+1)}{n}y\right)$ in equation (2.32) on both sides, that

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)$$
$$+f\left(-\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+2f\left(\frac{n}{n-1}x+\frac{(n+2)}{n+1}z\right)+2f\left(\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right) \quad (2.33)$$
$$+2f\left(\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y\right)=8\left(\frac{n}{n-1}\right)^2 f(x)$$
$$+8\left(\frac{(n+1)}{n}\right)^2 f(y)+12\left(\frac{(n+2)}{n+1}\right)^2 f(z)+2f\left(\frac{(n+1)}{n}y\right)$$

for all $x, y, z \in X$. But we know that from the equation (2.33), we arrive

$$f\left(\frac{(n+1)}{n}y+\frac{n}{n-1}x+\frac{(n+2)}{n+1}z\right)+f\left(\frac{(n+1)}{n}y-\left(\frac{n}{n-1}x+\frac{(n+2)}{n+1}z\right)\right)$$
$$=2f\left(\frac{(n+1)}{n}y\right)+2f\left(\frac{n}{n-1}x+\frac{(n+2)}{n+1}z\right) \quad (2.34)$$

for all $x, y, z \in X$. Substituting the value of (2.34) in (2.33), we obtain

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)$$
$$+f\left(-\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(-\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)$$

$$+f\left(\frac{(n+1)}{n}y+\frac{n}{n-1}x+\frac{(n+2)}{n+1}z\right)+f\left(\frac{(n+1)}{n}y-\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)$$
$$+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)$$
$$=8\left(\frac{n}{n-1}\right)^2 f(x)+8\left(\frac{(n+1)}{n}\right)^2 f(y)+12\left(\frac{(n+2)}{n+1}\right)^2 f(z)+2\left(\frac{(n+1)}{n}\right)^2 f(y) \tag{2.35}$$

for all $x, y, z \in X$. Adding $2f\left(\frac{(n+1)}{n}y\right)$ in equation (5.2), on both sides, we have

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)$$
$$+f\left(-\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{(n+1)}{n}y+\frac{n}{n-1}x+\frac{(n+2)}{n+1}z\right)+f\left(\frac{(n+1)}{n}y-\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right) \tag{2.36}$$
$$+2f\left(\frac{(n+1)}{n}y\right)+2f\left(\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)$$
$$=8\left(\frac{n}{n-1}\right)^2 f(x)+8\left(\frac{(n+1)}{n}\right)^2 f(y)+12\left(\frac{(n+2)}{n+1}\right)^2 f(z)+4\left(\frac{(n+1)}{n}\right)^2 f(y)$$

for all $x, y, z \in X$. From the equation (2.36) remodifying, that

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)$$
$$+f\left(-\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right) \tag{2.37}$$
$$+f\left(\frac{(n+1)}{n}y+\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)+f\left(\frac{(n+1)}{n}y-\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)$$
$$+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)=8\left(\frac{n}{n-1}\right)^2 f(x)+12\left(\frac{(n+1)}{n}\right)^2 f(y)+12\left(\frac{(n+2)}{n+1}\right)^2 f(z)$$

for all $x, y, z \in X$. Adding $2f\left(\frac{n}{n-1}x\right)$ in equation (2.37), on both sides, we obtain

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)$$
$$+f\left(-\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)$$
$$+f\left(\frac{(n+1)}{n}y+\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)+f\left(\frac{(n+1)}{n}y-\frac{n}{n-1}x-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{n}{n-1}x\right)+2f\left(\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right) \tag{2.38}$$
$$+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)=8\left(\frac{n}{n-1}\right)^2 f(x)+2f\left(\frac{n}{n-1}x\right)+12\left(\frac{(n+1)}{n}\right)^2 f(y)+12\left(\frac{(n+2)}{n+1}\right)^2 f(z)$$

for all $x, y, z \in X$. From the equation (2.38) and remodifying, we arrive

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=2f\left(\frac{n}{n-1}x\right)+2f\left(\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right) \tag{2.39}$$

for all $x, y, z \in X$. Substituting the value of (2.39) in (2.38), we get

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)$$
$$+f\left(-\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right) \tag{2.40}$$
$$+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)=10\left(\frac{n}{n-1}\right)^2 f(x)$$
$$+12\left(\frac{(n+1)}{n}\right)^2 f(y)+12\left(\frac{(n+2)}{n+1}\right)^2 f(z)+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)=10\left(\frac{n}{n-1}\right)^2 f(x)$$
$$+12\left(\frac{(n+1)}{n}\right)^2 f(y+12\left(\frac{(n+2)}{n+1}\right)^2 f(z)$$

for all $x, y, z \in X$. Adding $2f\left(\frac{n}{n-1}x\right)$ in equation (5.7), on both sides, we get

$$\begin{aligned}
&f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)\\
&+f\left(-\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(-\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)\\
&+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)\\
&+f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)\\
&+2f\left(\frac{n}{n-1}x\right)+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)=10\left(\frac{n}{n-1}\right)^2 f(x)+12\left(\frac{(n+1)}{n}\right)^2 f(y)+12\left(\frac{(n+2)}{n+1}\right)^2 f(z)
\end{aligned} \tag{2.41}$$

for all $x, y, z \in X$. From the equation (2.41) remodifying, we arrive

$$f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)=2f\left(\frac{n}{n-1}x\right)+2f\left(\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right) \tag{2.42}$$

for all $x, y, z \in X$. Substituting the value of (2.42) in (2.41), we obtain

$$\begin{aligned}
&f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)\\
&+f\left(-\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)+f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)\\
&+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(-\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)\\
&+f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x+\frac{(n+1)}{n}y-\frac{(n+2)}{n+1}z\right)+f\left(\frac{n}{n-1}x-\frac{(n+1)}{n}y+\frac{(n+2)}{n+1}z\right)\\
&\qquad=12\left(\frac{n}{n-1}\right)^2 f(x)+12\left(\frac{(n+1)}{n}\right)^2 f(y)+12\left(\frac{(n+2)}{n+1}\right)^2 f(z)
\end{aligned} \tag{2.43}$$

for all $x, y, z \in X$ .Using eveness from (2.43) and then desired our result.

Conversely, $f : X \to Y$ satisfies the functional equation (3.5) with $f(0) = 0$. For assume that $f : X \to Y$ satisfies the functional equation (6.4) with $f(0) = 0$. Replacing (x, y, z) by $(x,0,0)$, $(0, x,0)$ and $(0,0, x)$, respectively in (6.4), we obtain

$$f\left(\frac{n}{n-1}x\right)=\left(\frac{n}{n-1}\right)^2 f(x);\ f\left(\frac{n+1}{n}y\right)=\left(\frac{n+1}{n}\right)^2 f(y) \text{ and } f\left(\frac{n+2}{n+1}z\right)=\left(\frac{n+2}{n+1}\right)^2 f(z) \tag{2.46}$$

for all $x \in X$. If f is an even function. Replacing (x, y, z) by $\left(\frac{n-1}{n}x, \frac{n+1}{n}y, 0\right)$ in (1.3), we get the result desired.

3. Additive Stability results (1.3)

Theorem 3.1: Let $j \in \{-1,1\}$ and $\alpha : X^3 \to [0,\infty)$ be a function such that

$$\sum_{k=0}^{\infty}\frac{\alpha\left(n^{kj}x, n^{kj}y, n^{kj}z\right)}{n^{kj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{k\to\infty}\frac{\alpha\left(n^{kj}x, n^{kj}y, n^{kj}z\right)}{n^{kj}}=0 \tag{3.1}$$

for all $x, y, z \in X$. Let $f_a : X \to Y$ be an odd function satisfying the inequality

$$\| Df_a(x, y, z) \| \le \alpha(x, y, z) \tag{3.2}$$

for all $x, y, z \in X$. There exists a unique additive mapping $A: X \to Y$ which satisfies the functional equation (1.3) and

$$\left\| f(x)-A(x)\right\| \le \frac{n-1}{2n}\sum_{k=0}^{\infty}\frac{(n-1)^k}{n^k}\alpha\left(\frac{n^k x}{(n-1)^k},0,0\right) \tag{3.3}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x)=\lim_{k\to\infty}\left(\frac{n-1}{n}\right)^k f\left(\left(\frac{n}{n-1}\right)^k x\right) \tag{3.4}$$

for all $x \in X$.

Proof. Assume that $j=1$. Putting (x,y,z) and $(x,0,0)$ in (3.2) and using oddness of f_a, we get

$$\left\| 2 f_a\left(\frac{n}{n-1}x\right) - 2\frac{n}{n-1} f(x) \right\| \le \alpha(x,0,0) \tag{3.5}$$

for all $x \in X$. Dividing $\frac{2n}{n-1}$ in (3.5) on both sides, we obtain

$$\left\| f_a(x) - \frac{n-1}{n} f_a\left(\frac{n}{n-1}x\right) \right\| \le \frac{n-1}{2n}\alpha(x,0,0) \tag{3.6}$$

for all $x \in X$. Replacing x by $\left(\frac{n}{n-1}x\right)$ in (3.6), we get

$$\left\| f_a\left(\frac{n}{n-1}x\right) - \frac{n-1}{n} f_a\left[\left(\frac{n}{n-1}\right)^2 x\right] \right\| \le \frac{n-1}{2n}\alpha\left(\frac{n}{n-1}x,0,0\right) \tag{3.7}$$

for all $x \in X$. Dividing by $\left(\frac{n}{n-1}x\right)$ in (3.7), we arrive

$$\left\| \frac{n-1}{n} f_a\left(\frac{n}{n-1}x\right) - \left(\frac{n-1}{n}\right)^2 f_a\left[\left(\frac{n}{n-1}\right)^2 x\right] \right\| \le \frac{(n-1)^2}{2n^2}\alpha\left(\frac{n}{n-1}x,0,0\right) \tag{3.8}$$

for all $x \in X$. It is follows from (3.6) and (3.8) we get

$$\left\| f_a(x) - \left(\frac{n-1}{n}\right)^2 f_a\left[\left(\frac{n}{n-1}\right)^2 x\right] \right\| \le \frac{n-1}{2n}\left[\alpha(x,0,0) + \frac{n-1}{2n}\alpha\left(\frac{n}{n-1}x,0,0\right)\right] \tag{3.9}$$

for all $x \in X$. In general for any positive integer k such that

$$\left\| f_a(x) - \left(\frac{n-1}{n}\right)^k f_a\left[\left(\frac{n}{n-1}\right)^k x\right] \right\| \le \sum_{k=0}^{n-1}\alpha\left(\frac{n^k x}{n-1},0,0\right)\cdot\frac{n-1}{2n}$$

$$\le \frac{n-1}{2n}\sum_{k=0}^{\infty}\frac{\alpha\left(\frac{n^k x}{(n-1)^k},0,0\right)}{\frac{n^k}{(n-1)^k}}$$

$$\le \frac{n-1}{2n}\sum_{k=0}^{\infty}\frac{(n-1)^k}{n^k}\alpha\left(\frac{n^k x}{(n-1)^k},0,0\right) \tag{3.10}$$

for all $x \in X$. In order to prove convergence of the sequence

$$\left\{ f_a\left(\frac{n}{n-1}\right)^k \left(\frac{n-1}{n}\right)^k \right\},$$

for all $x \in X$. Replacing x by $\left(\frac{n}{n-1}\right)^l x$ and dividing $\left(\frac{n}{n-1}\right)^l$ in (3.10), for any $k,l>0$, to deduce

$$\left\| \left(\frac{n-1}{n}\right)^l f_a\left[\left(\frac{n}{n-1}\right)^l x\right] - \left(\frac{n-1}{n}\right)^{k+l} f_a\left[\left(\frac{n}{n-1}\right)^{k+l} x\right] \right\| \tag{3.11}$$

$$\le \frac{n-1}{2n}\sum_{k=0}^{\infty}\left(\frac{n-1}{n}\right)^{k+l}\alpha\left(\left(\frac{n}{n-1}\right)^{k+l}x,0,0\right)$$

$\to 0$ as $l \to \infty$

for all $x \in X$ .Hence the sequence $\left\{ f_a\left(\frac{n}{n-1}\right)^k x\cdot\left(\frac{n-1}{n}\right)^k \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X \to Y$ such that

$$A(x) = \lim_{k\to\infty} f_a\left(\frac{n}{n-1}\right)^k x\cdot\left(\frac{n-1}{n}\right)^k, \qquad \forall\ x \in X . \tag{3.12}$$

Letting $k \to \infty$ in (3.12), we see that (3.3) holds for $x \in X$. To prove that A satisfies (1.1) replacing (x,y,z) by $\left(\left(\frac{n}{n-1}\right)^l x, \left(\frac{n}{n-1}\right)^l y, \left(\frac{n}{n-1}\right)^l z\right)$ and dividing $\left(\frac{n}{n-1}\right)^l$ in (3.2), we obtain

$$\left(\frac{n-1}{n}\right)^{l}\left\|Df_{a}\left[\left(\frac{n}{n-1}\right)^{l}x,\left(\frac{n}{n-1}\right)^{l}y,\left(\frac{n}{n-1}\right)^{l}z\right]\le\frac{n-1}{2n}\alpha\left(\left(\frac{n}{n-1}\right)^{l}x,\left(\frac{n}{n-1}\right)^{l}y,\left(\frac{n}{n-1}\right)^{l}z\right)\right.$$

for all $x, y, z \in X$. Letting $k \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$D_A(x, y, z) = 0$.

Hence A satisfies (1.1) for all $x, y, z \in X$. To show that A is unique, let $B(x)$ be another additive mapping satisfying (1.1) and (3.3), then

$$\| A(x) - B(x) \| = \left(\frac{n-1}{n}\right)^{l}\left\| A\left(\frac{n}{n-1}\right)^{l} x - B\left(\frac{n}{n-1}\right)^{l} x\right\|$$

$$\le \left(\frac{n-1}{n}\right)^{l}\left\{\left\| A\left(\frac{n}{n-1}\right)^{l} x - f_a\left(\frac{n}{n-1}\right)^{l} x\right\| + \left\| f_a\left(\frac{n}{n-1}\right)^{l} x - B\left(\frac{n}{n-1}\right)^{l} x\right\|\right\}$$

$$\le \frac{n-1}{2n}\sum_{k=0}^{\infty}\left(\frac{n-1}{n}\right)^{k+l}\alpha\left(\left(\frac{n}{n-1}\right)^{k+l} x, 0, 0\right)$$

$\to 0$ as $l \to \infty$

for all $x \in X$. Hence A is unique. For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.1).

Corollary 3.2: Let p and t be a nonnegative real numbers. Let an odd function $f_a : X \to Y$ satisfying the inequality

$$\| D_a(x, y, z) \| \le \begin{cases} p \\ p\left(\| x \|^{t} + \| y \|^{t} + \| z \|^{t}\right), & s < 1 \text{ or } s > 1 \\ p\left(\| x \|^{t} \| y \|^{t} \| z \|^{t} + \left\{\| x \|^{3t} + \| y \|^{3t} + \| z \|^{3t}\right\}\right), & s < \frac{1}{3} \text{ or } s > \frac{1}{3} \end{cases}$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\| f_a(x) - A(x) \| \le \begin{cases} \frac{p}{2}(n-1), \\ \frac{p}{2}\left[\frac{1}{\left(\frac{n}{n-1}\right) - \left(\frac{n}{n-1}\right)^{t}}\right]\| x \|^{t}, \\ \frac{p}{2}\left[\frac{1}{\left(\frac{n}{n-1}\right) - \left(\frac{n}{n-1}\right)^{3t}}\right]\| x \|^{3t}, \end{cases}$$

for all $x \in X$.

4. Quadratic stability result for (1.3)

Theorem 4.1: Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function such that

$$\sum_{k=0}^{\infty}\frac{\alpha\left(n^{kj}x, n^{kj}y, n^{kj}z\right)}{n^{2kj}} \text{ converges in } \mathbb{R} \text{ and } \lim_{k\to\infty}\frac{\alpha\left(n^{kj}x, n^{kj}y, n^{kj}z\right)}{n^{2kj}} = 0 \tag{4.1}$$

for all $x, y, z \in X$. Let $f_q : X \to Y$ be an even function satisfying the inequality

$$\left\| Df_q(x, y, z) \right\| \le \alpha(x, y, z) \tag{4.2}$$

for all $x, y, z \in X$. There exists a unique quadratic mapping $Q : X \to Y$ such that,

$$\| f_q(x) - Q(x) \| \le\le \frac{1}{4}\left(\frac{n-1}{n}\right)^{2}\sum_{k=0}^{\infty}\left(\frac{n-1}{n}\right)^{2k}\alpha\left(\left(\frac{n}{n-1}\right)^{k} x, 0, 0\right) \tag{4.3}$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k \to \infty} \left(\frac{n-1}{n} \right)^{2k} f\left(\left(\frac{n}{n-1} \right)^{k} x \right) \quad (4.4)$$

for all $x \in X$.

Proof. Assume that $j = 1$. Setting (x, y, z) and $(x, 0, 0)$ in (4.2) and using evenness of f_q, we get

$$\left\| 4 f_q\left(\frac{n}{n-1} x \right) - 4 \frac{n}{n-1} f_q(x) \right\| \le \alpha(x, 0, 0) \quad (4.5)$$

for all $x \in X$, dividing $4\left(\frac{n}{n-1} \right)$ in (4.5) on both sides, we obtain

$$\left\| \left(\frac{n-1}{n} \right)^2 f_q\left(\frac{n}{n-1} x \right) - f_q(x) \right\| \le \frac{1}{4} \cdot \left(\frac{n-1}{n} \right)^2 \alpha(x, 0, 0) \quad (4.6)$$

for all $x \in X$. Replacing x by $\left(\frac{n}{n-1} x \right)$ in (4.6), we get

$$\left\| \left(\frac{n-1}{n} \right)^2 f_q\left(\left(\frac{n}{n-1} \right)^2 x \right) - f_q\left(\frac{nx}{n-1} \right) \right\| \le \frac{1}{4} \cdot \left(\frac{n-1}{n} \right)^2 \alpha\left(\frac{nx}{n-1}, 0, 0 \right) \quad (4.7)$$

for all $x \in X$, dividing by $\left(\frac{n-1}{n} \right)^2$ in (4.7), that

$$\left\| \left(\frac{n-1}{n} \right)^4 f_q\left(\left(\frac{n}{n-1} \right)^2 x \right) - \left(\frac{n-1}{n} \right)^2 f_q\left(\frac{nx}{n-1} \right) \right\| \le \frac{1}{4} \cdot \left(\frac{n-1}{n} \right)^4 \alpha\left(\frac{nx}{n-1}, 0, 0 \right) \quad (4.8)$$

for all $x \in X$. It is follows from (4.6) and (4.8) we get

$$\left\| \left(\frac{n-1}{n} \right)^4 f_q\left(\left(\frac{n}{n-1} \right)^2 x \right) - f_q(x) \right\| \le \frac{1}{4} \cdot \left(\frac{n-1}{n} \right)^2 \left[\alpha(x, 0, 0) + \left(\frac{n-1}{n} \right)^2 \alpha\left(\frac{nx}{n-1}, 0, 0 \right) \right] \quad (4.9)$$

for all $x \in X$. In general for any positive integer k such that

$$\left\| f_q(x) - \left(\frac{n-1}{n} \right)^{2k} f_q\left[\left(\frac{n}{n-1} \right)^k x \right] \right\| \le \sum_{k=0}^{n-1} \alpha\left(\frac{n^k x}{n-1}, 0, 0 \right) \cdot \frac{1}{4} \left(\frac{n-1}{n} \right)^2$$

$$\le \frac{1}{4} \left(\frac{n-1}{n} \right)^2 \sum_{k=0}^{\infty} \left(\frac{n-1}{n} \right)^{2k} \alpha\left(\left(\frac{n}{n-1} \right)^k x, 0, 0 \right) \quad (4.10)$$

for all $x \in X$. In order to prove convergence of the sequence

$$\left\{ \left(\frac{n-1}{n} \right)^{2k} f_q\left(\frac{n}{n-1} \right)^k x \right\},$$

for all $x \in X$. Replacing x by $\left(\frac{n}{n-1} \right)^l x$ and dividing $\left(\frac{n}{n-1} \right)^{2l}$ for any $k, l > 0$, to deduce

$$\left\| f_q\left(\frac{n}{n-1} \right)^l x - \left(\frac{n-1}{n} \right)^{2k} f_q\left[\left(\frac{n}{n-1} \right)^k \cdot \left(\frac{n}{n-1} \right)^l x \right] \right\|$$

$$\le \frac{1}{4} \left(\frac{n-1}{n} \right)^2 \sum_{k=0}^{\infty} \left(\frac{n-1}{n} \right)^{2(k+l)} \alpha\left(\left(\frac{n}{n-1} \right)^{k+l} x, 0, 0 \right) \quad (4.11)$$

$\to 0$ as $l \to \infty$

for all $x \in X$ .Hence the sequence $\left\{ \left(\frac{n-1}{n} \right)^{2k} f_q\left(\frac{n}{n-1} \right)^k x \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $Q : X \to Y$ such that

$$Q(x) = \lim_{k \to \infty} \left\{ \left(\frac{n-1}{n} \right)^{2k} f_q\left(\frac{n}{n-1} \right)^k x \right\}, \ \forall \ x \in X . \quad (4.12)$$

Letting $k \to \infty$ in (4.11), we see that (4.3) holds for $x \in X$. To prove that Q satisfies (1.2) replacing (x, y, z) by $\left(\left(\frac{n}{n-1} \right)^l x, \left(\frac{n}{n-1} \right)^l y, \left(\frac{n}{n-1} \right)^l z \right)$ and dividing $\left(\frac{n}{n-1} \right)^{2l}$ in (4.2), that

$$\left(\frac{n-1}{n}\right)^{2l}\left\|Df_q\left[\left(\frac{n}{n-1}\right)^l x,\left(\frac{n}{n-1}\right)^l y,\left(\frac{n}{n-1}\right)^l z\right]\le\frac{1}{4}\left(\frac{n-1}{n}\right)^2\sum_{k=0}^{\infty}\alpha\left(\left(\frac{n}{n-1}\right)^l x,\left(\frac{n}{n-1}\right)^l y,\left(\frac{n}{n-1}\right)^l z\right)\right.$$

for all $x, y, z \in X$. Letting $k \to \infty$ in the above inequality and using the definition of $Q(x)$, we see that

$D_Q(x, y, z) = 0$.

Hence Q satisfies (1.2) for all $x, y, z \in X$. To show that Q is unique, let $B(x)$ be another quadratic mapping satisfying (1.1) and (4.3), then

$$\| Q(x)-B(x) \| = \left(\frac{n-1}{n}\right)^l \left\| Q\left(\frac{n}{n-1}\right)^l x - B\left(\frac{n}{n-1}\right)^l x \right\|$$

$$\le \left(\frac{n-1}{n}\right)^{2l}\left\{\left\| Q\left(\frac{n}{n-1}\right)^l x - f_q\left(\frac{n}{n-1}\right)^l x\right\| + \left\| f_q\left(\frac{n}{n-1}\right)^l x - B\left(\frac{n}{n-1}\right)^l x\right\|\right\}$$

$$\le \frac{1}{4}\left(\frac{n-1}{n}\right)^2\sum_{k=0}^{\infty}\left(\frac{n-1}{n}\right)^{2(k+l)}\alpha\left(\left(\frac{n}{n-1}\right)^{k+l}x,0,0\right)$$

$\to 0$ as $l \to \infty$

for all $x \in X$. Hence Q is unique. For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 4.1 concerning the stability of (1.1).

Corollary 4.1: Let p and t be a nonnegative real numbers. Let an even function $f_q : X \to Y$ satisfying the inequality

$$\| D_q(x,y,z) \| \le \begin{cases} p \\ p\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & s<1 \text{ or } s>1 \\ p\left(\|x\|^t\|y\|^t\|z\|^t + \left\{\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t}\right\}\right), & s<\frac{1}{3} \text{ or } s>\frac{1}{3} \end{cases}$$

for all $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\| f_q(x)-Q(x) \| \le \begin{cases} \frac{p}{4}\left(\frac{n}{n-1}\right)^2, \\ \frac{p}{4}\|x\|^t\left(\frac{1}{\left(\frac{n}{n-1}\right)^2-\left(\frac{n}{n-1}\right)^t}\right), \\ \frac{p}{4}\|x\|^{3t}\left(\frac{1}{\left(\frac{n}{n-1}\right)^2-\left(\frac{n}{n-1}\right)^{3t}}\right), \end{cases}$$

for all $x \in X$.

5. Additive Quadratic Stability results (1.3)

Theorem 5.1: Let $j \in \{-1,1\}$ and $\alpha : X^3 \to [0,\infty)$ be a function satisfying (3.2) and (4.2) for all $x, y, z \in X$. Let $f : X \to Y$ be a function satisfying the inequality

$$\| Df(x,y,z) \| \le \alpha(x,y,z) \qquad (4.13)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x)-A(x)-Q(x) \| \le \frac{1}{2}\left[\frac{n-1}{2n}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(\left(\frac{n}{n-1}\right)^{kj}x,0,0)}{\left(\frac{n}{n-1}\right)^{kj}}+\frac{\alpha(\left(-\frac{n}{n-1}\right)^{kj}x,0,0)}{\left(\frac{n}{n-1}\right)^{kj}}\right)\right.$$

$$\left.+\frac{1}{4}\left(\frac{n-1}{n}\right)^2\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(\left(\frac{n}{n-1}\right)^{2kj}x,0,0)}{\left(\frac{n}{n-1}\right)^{2kj}}+\frac{\alpha(-\left(\frac{n}{n-1}\right)^{2kj}x,0,0)}{\left(\frac{n}{n-1}\right)^{2kj}}\right)\right] \qquad (4.14)$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ is defined in (3.4) and (4.4) respectively for all $x \in X$.

Proof. Let $f_o(x) = \dfrac{f_a(x) - f_a(-x)}{2}$ for all $x \in X$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in X$. Hence,

$$\| Df_o(x, y, z) \| \le \frac{\alpha(x, y, z)}{2} + \frac{\alpha(-x, -y, -z)}{2} \tag{4.15}$$

for all $x, y, z \in X$. By Theorem, we have

$$\| f_o(x) - A(x) \| \le \frac{n-1}{2n} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha\left(\left(\frac{n}{n-1}\right)^{kj} x, 0, 0\right)}{\left(\frac{n}{n-1}\right)^{kj}} + \frac{\alpha\left(-\left(\frac{n}{n-1}\right)^{kj} x, 0, 0\right)}{\left(\frac{n}{n-1}\right)^{kj}} \right) \tag{4.16}$$

For all $x \in X$. Also let, $f_e(x) = \dfrac{f_q(x) + f_q(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in X$. Hence,

$$\| Df_e(x, y, z) \| \le \frac{\alpha(x, y, z)}{2} + \frac{\alpha(-x, -y, -z)}{2} \tag{4.17}$$

for all $x, y, z \in X$. By Theorem, we have

$$\| f_e(x) - Q(x) \| \le \frac{1}{4}\left(\frac{n-1}{n}\right)^2 \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(n^{kj}x, 0, 0)}{n^{2kj}} + \frac{\alpha(-n^{kj}x, 0, 0)}{n^{2kj}} \right) \tag{4.18}$$

for all $x \in X$. Define

$$f(x) = f_e(x) + f_o(-x) \tag{4.19}$$

for all $x \in X$. It follows from (4.16) and (4.18), we arrive

$$\| f(x) - A(x) - Q(x) \| = \| f_e(x) + f_o(-x) - A(x) - Q(x) \|$$

$$\le \| f_o(-x) - A(x) \| + \| f_e(x) - Q(x) \|$$

$$\| f(x) - A(x) - Q(x) \| \le \frac{1}{2} \left[\frac{n-1}{2n} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(\left(\frac{n}{n-1}\right)^{kj} x, 0, 0)}{\left(\frac{n}{n-1}\right)^{kj}} + \frac{\alpha(\left(-\frac{n}{n-1}\right)^{kj} x, 0, 0)}{\left(\frac{n}{n-1}\right)^{kj}} \right) \right.$$

$$\left. + \frac{1}{4}\left(\frac{n-1}{n}\right)^2 \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(\left(\frac{n}{n-1}\right)^{2kj} x, 0, 0)}{\left(\frac{n}{n-1}\right)^{2kj}} + \frac{\alpha(-\left(\frac{n}{n-1}\right)^{2kj} x, 0, 0)}{\left(\frac{n}{n-1}\right)^{2kj}} \right) \right]$$

for all $x \in X$. Hence the theorem is proved.

Corollary 4.2: *Let p and t be a nonnegative real numbers. Let a function $f : X \to Y$ satisfying the inequality*

$$\| D_f(x, y, z) \| \le \begin{cases} \lambda \\ \lambda\left(\| x \|^t + \| y \|^t + \| z \|^t\right), \\ \lambda\left(\| x \|^t \| y \|^t \| z \|^t + \left\{\| x \|^{3t} + \| y \|^{3t} + \| z \|^{3t}\right\}\right), \end{cases}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and unique quadratic function $Q : X \to Y$ such that

$$\| f_q(x) - A(x) - Q(x) \| \le \begin{cases} \frac{p}{2}\left(|n-1| + \frac{1}{2}\left|\left(\frac{n-1}{n}\right)^2\right| \right), \\ \frac{p\| x \|^t}{2}\left(\frac{1}{\left|\left(\frac{n}{n-1}\right) - \left(\frac{n}{n-1}\right)^t\right|} - \frac{1}{2\left|\left(\frac{n}{n-1}\right)^2 - \left(\frac{n}{n-1}\right)^t\right|} \right), \\ \frac{p\| x \|^{3t}}{2}\left(\frac{1}{\left|\left(\frac{n}{n-1}\right) - \left(\frac{n}{n-1}\right)^{3t}\right|} - \frac{1}{2\left|\left(\frac{n}{n-1}\right)^2 - \left(\frac{n}{n-1}\right)^{3t}\right|} \right), \end{cases}$$

for all $x \in X$.

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