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Stability of quadratic functional equations

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Abstract
In this paper, the authors proved the generalized Ulam-Hyers stability of quadratic functional equation of the form
$$f(x+y+3z)+f(x+y-3z)=9f(x+y+z)+9f(x+y-z)-16f(x+y)$$
In Banach space using direct method.

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1. Introduction

A classical question in the theory of functional equation is the following: when is it true that a function which approximately satisfies a functional equation ε must be close to an exact solution of ε ?"

If the problem accepts a solution, we say that the question ε is stable. The first stability problem concerning group homomorphisms was raised by Ulam [23] in 1940. We are a group G and metric group G' with metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that $f: G \to G'$ satisfies $d\left(f(xy), f(x)f(y)\right) < \delta$, for all $x, y \in G$, then a homomorphism $h: G \to G'$; exists with $d\left(f(x), h(x)\right) < \varepsilon$ for all $x \in G$?

In the next year D. H. Hyers [8], gave a positive answer, to the above question for additive groups under the assumption that the groups are Banach spaces.

These terminologies are also applied to the case of other functional equations and it has been extensively investgated by a number of authors and there are many interesting results concering this problem including quadratic functional equations (see and references cited therein) [3-10, 14, 18].

In this paper, the authors investgate the generalized Ulam-Hyers stability of a quadratic functional equation

$$f(x+y+3z)+f(x+y-3z)=9f(x+y+z)+9f(x+y-z)-16f(x+y) \qquad (1.1)$$

In Banach spaces.

In section 2, the generalized Ulam-Hyers stability of the functional equation (1.1) is proved.

In section 3, the generalized Ulam-Hyers stability of generalized quadratic functional equation (1.1) is investigated by using another substitution. Hence the details of the poof are omitted. Here after, throughout this paper, let us consider X and Y to be a normed space and Banach, respectively. Define a mapping

$$Df\left(x,y,z\right)= f(x+y+3z)+f(x+y-3z)-9f(x+y+z)-9f(x+y-z)+16f(x+y)$$

For all $x, y, z \in X$

2. Stability Results for (1.1)

In this section, we present the generalized Ulam-Hyers stability of the functional equation (1.1) for even case.

Theorem 2.1. *Let* $j \in \{-1,1\}$ *and* $\alpha : X^3 \to [0,\infty)$ *be an even function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(3^{kj}x, 3^{kj}y, 3^{kj}z\right)}{9^{kj}} \text{ Converges in } \mathbb{R} \text{ and } \lim_{k\to\infty} \frac{\alpha\left(3^{kj}x, 3^{kj}y, 3^{kj}z\right)}{9^{kj}} = 0 \tag{2.1}$$

For all $x, y, z \in X$ *. Let* $f_q : X \to Y$ *be an even function satisfying the inequality*

$$\left\| Df_q(x,y,z) \right\| \le \alpha(x,y,z) \tag{2.2}$$

For all $x, y, z \in X$ *. There exists a unique quadratic mapping* $Q : X \to Y$ *which satisfies the functional equation* (1.1) *and*

$$\left\| f_q(x) - Q(x) \right\| \le \frac{1}{18} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(0,0,3^{kj}x\right)}{9^{kj}} \tag{2.3}$$

For all $x \in X$ *. The mapping* $Q(x)$ *is defined by*

$$Q(x) = \lim_{k\to\infty} \frac{f_q\left(3^{kj}x\right)}{9^{kj}} \tag{2.4}$$

For all $x \in X$ *.*

Proof. Assume that $j = 1$. Replacing (x,y,z) and $(0,0,x)$ in (2.2) of f_q, we get

$$\left\| 2f_q(3x) - 18f_q(x) \right\| \le \alpha(0,0,x) \tag{2.5}$$

For all $x \in X$. It follows from (2.5) that

$$\left\| \frac{f_q(3x)}{9} - f_q(x) \right\| \le \frac{\alpha}{18}(0,0,x) \tag{2.6}$$

For all $x \in X$. Replacing x by $3x$ in (2.6) and dividing by 9, we obtain

$$\left\| \frac{f_q(3^2x)}{9^2} - \frac{f_q(3x)}{9} \right\| \le \frac{\alpha}{18}\frac{(0,0,3x)}{9} \tag{2.7}$$

For all $x \in X$. It follows from (2.5) and (2.6) that

$$\left\| \frac{f_q(3^2x)}{3^2} - f_q(x) \right\| \le \frac{1}{18}\left[\alpha(0,0,x) + \frac{\alpha}{9}(0,0,3x)\right] \tag{2.8}$$

For all $x \in X$. Generalizing, we have

$$\left\| f_q(x) - \frac{f_q(3^kx)}{9^k} \right\| \le \frac{1}{18}\sum_{k=0}^{n-1}\frac{\alpha\left(0,0,3^kx\right)}{9^k} \le \frac{1}{18}\sum_{k=0}^{\infty}\frac{\alpha\left(0,0,3^kx\right)}{9^k} \tag{2.9}$$

for all $x \in X$. In order to prove convergence of the sequence

$$\left\{ \frac{f_q(3^kx)}{9^k} \right\},$$

Replace x by $3^l x$ and dividing 3^l (2.9), for any $k, l > 0$, to deduce

$$\left\| \frac{f_q(3^lx)}{9^l} - \frac{f_q(3^{k+l}x)}{9^{k+l}} \right\| = \frac{1}{9^l}\left\| f_q(3^lx) - \frac{f_q(3^k.3^lx)}{9^k} \right\|$$

$$\le \frac{1}{18}\sum_{k=0}^{n-1}\frac{\alpha\left(0,0,3^{k+l}x\right)}{9^{k+l}}$$

$$\le \frac{1}{18}\sum_{k=0}^{\infty}\frac{\alpha\left(0,0,3^{k+l}x\right)}{9^{k+l}} \tag{2.10}$$

$\to 0$ as $l \to \infty$

for all $x \in X$.

Hence the sequence $\left\{ \frac{f_q(3^kx)}{9^k} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $Q : X \to Y$ such that

$$\mathrm{Q}(x)=\lim_{k\to\infty}\frac{f_q(3^k x)}{9^k}, \quad \forall\ x\in X.$$

Letting $k\to\infty$ in (2.9), we see that (2.3) holds for $x\in X$. To prove that Q satisfies (1.1) replacing (x,y,z) by $(3^k x,3^k y,3^k z)$ and dividing 9^k in (2.2), we obtain

$$\frac{1}{9^k}\left\| Df_q\left(3^k x,3^k y,3^k z\right)\right\|\le\frac{1}{9^k}\alpha\left(3^k x,3^k y,3^k z\right)$$

for all $x,y,z\in X$. Letting $k\to\infty$ in the above inequality and using the definition of $\mathrm{Q}(x)$, we see that $DQ(x,y,z)=0$ Hence Q satisfies (1.1) for all $x,y,z\in X$. To show that Q is unique, let $B(x)$ be another quadratic mapping satisfying (1.1) and (2.3), then

$$\| Q(x)-B(x)\|=\frac{1}{9^l}\| Q(3^l x)-B(3^l x)\|\le\frac{1}{9^l}\left\| Q(3^l x)-f_q(3^l x)\right\|+\left\| f_q(3^l x)-B(3^l x)\right\|$$

$$\le\frac{1}{18}\sum_{k=0}^{\infty}\frac{\alpha\left(0,0,3^{k+l}x\right)}{9^{k+l}}\to 0 \text{ as } l\to\infty$$

for all $x\in X$. Hence Q is unique.

Now, replacing x by $\frac{x}{3}$ in (2.2), we get

$$\left\|2f_q(x)-18f_q\left(\frac{x}{3}\right)\right\|\le\alpha\left(0,0,\frac{x}{3}\right) \tag{2.11}$$

for all $x\in X$. It follows from (1.12) that

$$\left\| f_q(x)-9f_a\left(\frac{x}{3}\right)\right\|\le\frac{1}{2}\alpha\left(0,0,\frac{x}{3}\right) \tag{2.12}$$

for all $x\in X$. The rest of the proof is similar to that of $j=1$. Hence for $j=-1$ also the theorem is true. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 2.1 concerning the stability of (1.1).

Corollary 2.2. *Let γ and w be a nonnegative real numbers. Let an even function $f_q:X\to Y$ satisfying the inequality*

$$\| Df_q(x,y,z)\|\le\begin{cases}\gamma \\ \gamma\left(\| x\|^w+\| y\|^w+\| z\|^w\right), & s\ne 1; \\ \gamma\left(\| x\|^w\| y\|^w\| z\|^w+\left\{\| x\|^{3w}+\| y\|^{3w}+\| z\|^{3w}\right\}\right), & s\ne\frac{1}{3};\end{cases} \tag{2.13}$$

for all $x,y,z\in X$. Then there exists a unique quadratic function $\mathrm{Q}:X\to Y$ such that

$$\| f_q(x)-Q(x)\|\le\begin{cases}\frac{\gamma}{16}, \\ \frac{\gamma\| x\|^w}{2\left|9-3^w\right|}, \\ \frac{\gamma\| x\|^{3w}}{2\left|9-3^{3w}\right|},\end{cases} \tag{2.14}$$

for all $x\in X$.

Proof: If we replace

$$\alpha(x,y,z)=\begin{cases}\gamma;\\ \gamma\left(\|x\|^{w}+\|y\|^{w}+\|z\|^{w}\right);\\ \gamma\left(\|x\|^{w}\|y\|^{w}\|z\|^{w}+\left\{\|x\|^{3w}+\|y\|^{3w}+\|z\|^{3w}\right\}\right);\end{cases} \tag{2.15}$$

For all $x,y,z\in X$

3. Stability Results for (1.1): Using another Substitution-Direct Method

In this section, the generalized Uiam-Hyers stability of generalized quadratic functional equation (1.1) is investigated by using another substitution. Hence the details of the proof are omitted.

Theorem 3.1. *Let* $j\in\{-1,1\}$ *and* $\alpha: X^{3}\to[0,\infty)$ *be an even function such that*

$$\sum_{k=0}^{\infty}\frac{\alpha\left(4^{kj}x,4^{kj}y,4^{kj}z\right)}{16^{kj}} \text{ Converges in } \mathbb{R} \text{ and } \lim_{k\to\infty}\frac{\alpha\left(4^{kj}x,4^{kj}y,4^{kj}z\right)}{16^{kj}}=0 \tag{3.1}$$

For all $x,y,z\in X$ *. Let* $f_q: X\to Y$ *be an even function satisfying the inequality*

$$\left\| Df_q(x,y,z)\right\|\le\alpha(x,y,z) \tag{3.2}$$

For all $x,y,z\in X$ *. There exists a unique quadratic mapping* $\mathrm{Q}: X\to Y$ *which satisfies the functional equation* (1.1) *and*

$$\left\| f_q(x)-Q(x)\right\|\le\frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\alpha\left(4^{kj}x,0,4^{kj}x\right)}{16^{kj}} \tag{3.3}$$

For all $x\in X$ *. The mapping* $\mathrm{Q}(x)$ *is defined by*

$$\mathrm{Q}(x)=\lim_{k\to\infty}\frac{f_q\left(4^{kj}x\right)}{16^{kj}} \tag{3.4}$$

For all $x\in X$ *.*

Corollary 3.2. *Let* γ *and* w *be a nonnegative real numbers. Let an even function* $f_q: X\to Y$ *satisfying the inequality*

$$\left\| Df_q(x,y,z)\right\|\le\begin{cases}\gamma & \\ \gamma\left(\|x\|^{w}+\|y\|^{w}+\|z\|^{w}\right), & s\neq1;\\ \gamma\left(\|x\|^{w}\|y\|^{w}\|z\|^{w}+\left\{\|x\|^{3w}+\|y\|^{3w}+\|z\|^{3w}\right\}\right), & s\neq\frac{1}{3};\end{cases} \tag{3.5}$$

For all $x,y,z\in X$ *. Then there exists a unique quadratic function* $\mathrm{Q}: X\to Y$ *such that*

$$\left\| f_q(x)-Q(x)\right\|\le\begin{cases}\frac{\gamma}{15},\\ \frac{2\gamma\|x\|^{w}}{\left|16-4^{w}\right|},\\ \frac{2\gamma\|x\|^{3w}}{\left|16-4^{3w}\right|},\end{cases} \tag{3.6}$$

For all $x\in X$ *.*

Proof: If we replace

$$\alpha(x,y,z)=\begin{cases}\gamma;\\ \gamma\left(\|x\|^{w}+\|y\|^{w}+\|z\|^{w}\right);\\ \gamma\left(\|x\|^{w}\|y\|^{w}\|z\|^{w}+\left\{\|x\|^{3w}+\|y\|^{3w}+\|z\|^{3w}\right\}\right);\end{cases} \tag{3.7}$$

For all $x,y,z\in X$

Theorem 3.3. *Let* $j\in\{-1,1\}$ *and* $\alpha:X^{3}\to[0,\infty)$ *be a function such that*

$$\sum_{k=0}^{\infty}\frac{\alpha\left(5^{kj}x,5^{kj}y,5^{kj}z\right)}{25^{kj}}\ \text{Converges in } \mathbb{R} \ \text{ and } \sum_{k=0}^{\infty}\frac{\alpha\left(5^{kj}x,5^{kj}y,5^{kj}z\right)}{25^{kj}}=0 \tag{3.8}$$

For all $x,y,z\in X$ *. Let* $f_q:X\to Y$ *be an even function satisfying the inequality*

$$\left\| Df_q(x,y,z)\right\|\le\alpha(x,y,z) \tag{3.9}$$

For all $x,y,z\in X$ *. There exists a unique quadratic mapping* $\mathrm{Q}:X\to Y$ *which satisfies the functional equation* (1.1) *and*

$$\left\| f_q(x)-Q(x)\right\|\le\frac{1}{25}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\alpha\left(5^{kj}x,5^{kj}x,5^{kj}x\right)}{25^{kj}} \tag{3.10}$$

For all $x\in X$ *. The mapping* $\mathrm{Q}(x)$ *is defined by*

$$\mathrm{Q}(x)=\lim_{k\to\infty}\frac{f_q\left(5^{kj}x\right)}{25^{kj}} \tag{3.11}$$

For all $x\in X$ *.*

Corollary 3.4. *Let* γ *and* w *be a nonnegative real numbers. Let an even function* $f_q:X\to Y$ *satisfying the inequality*

$$\left\| Df_q(x,y,z)\right\|\le\begin{cases}\gamma\\ \gamma\left(\|x\|^{w}+\|y\|^{w}+\|z\|^{w}\right), & s\neq1;\\ \gamma\left(\|x\|^{w}\|y\|^{w}\|z\|^{w}+\left\{\|x\|^{3w}+\|y\|^{3w}+\|z\|^{3w}\right\}\right), & s\neq\frac{1}{3};\end{cases} \tag{3.12}$$

For all $x,y,z\in X$ *. Then there exists a unique quadratic function* $\mathrm{Q}:X\to Y$ *such that*

$$\left\| f_q(x)-Q(x)\right\|\le\begin{cases}\frac{\gamma}{24},\\ \frac{3\gamma\|x\|^{w}}{\left|25-5^{w}\right|},\\ \frac{3\gamma\|x\|^{3w}}{\left|25-5^{3w}\right|},\end{cases} \tag{3.13}$$

For all $x\in X$ *.*

Proof: If we replace

$$\alpha(x,y,z)=\begin{cases}\gamma;\\ \gamma\left(\|x\|^{w}+\|y\|^{w}+\|z\|^{w}\right);\\ \gamma\left(\|x\|^{w}\|y\|^{w}\|z\|^{w}+\left\{\|x\|^{3w}+\|y\|^{3w}+\|z\|^{3w}\right\}\right);\end{cases} \tag{3.14}$$

For all $x,y,z\in X$

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