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On ternary quadratic equation $x^2+ x y +y^2=12z^2$

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Abstract
The Ternary quadratic Diophantine equation given by $x^2+ xy +y^2=12z^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: Ternary, Quadratic, integral solutions, Polygonal Numbers.

1. Introduction
The ternary quadratic Diophantine equation offers an unlimited field for research because of their variely [1, 3]. For an extensive review of various problems, one may refer [1, 20]. This communication concerns with yet another interesting ternary quadratic equation $x^2+ xy + y^2 =12z^2$ for determining its infinitely many non-zero distinct integral solutions. Also a few interesting relations among the solutions have been xhibited.

2. Notations used:
- $t_{3,n}$-Polygonal number of rank n with size m
- P^3_n -Tetrahedral number of rank n
- P^4_n -Square pyramidal number of rank n
- P^5_n-Pentagonal pyramidal number of rank n

3. Method of Analysis
The Ternary Quadratic Diophantine Equation to be solved for its non-zero distinct integral solution is

$$x^2+ xy +y^2=12z^2 \qquad (1)$$

Pattern – I
On substitution of linear transformations ($u \neq v \neq 0$)

$$x=u+3v,\ y=u-3v \qquad (2)$$

In (1) leads to $u^2+ 3v^2 =4z^2$ (3)

The corresponding solutions of (3) is the form

$$\left.\begin{aligned} u&=a^2- 6ab-3b^2\\ v&=a^2+ 2ab-3b^2\\ z&=a^2+ 3b^2 \end{aligned}\right\} \qquad (4)$$

In view of (4), the solution of (1) can be written as

$$\left.\begin{aligned} x&=4a^2 - 12b^2\\ y&=6b^2 - 12ab - 2a^2\\ z&=a^2 + 3b^2 \end{aligned}\right\} \qquad (5)$$

Instead of (2) using the transformations x=u-3v, y=u+3v in (1), we get again (3) only, Thus, the integer solutions of (1) are obtained as

$$\left.\begin{aligned} x&=-2a^2 - 12ab + 6b^2\\ y&=4a^2 - 12b^2\\ z&=a^2 + 3b^2 \end{aligned}\right\} \qquad (6)$$

A few interesting properties observed are as follows:

I) $5x(2a, a) + y(2a, a) \equiv 0 \pmod 4$

II) $x(a, 1) - y(a,1) - 12t_{3,a} \equiv 0 \pmod 6$

III) $y(1, B) + 2(1, B) - t(64, B) + t(34, B) \equiv 0 \pmod{15}$

IV) $x(A, 2) - t_{202, A} + t_{198, A} \equiv -24 \pmod{26}$

Each of the following expression represents a nasty numbers.

$4z(a, b) - x(a, b)$

$3\{x(a, b) + 4z(a, b)\}$

$x(a, a) + 2y(a, a)$

$y(a, a) - 4z(a, a)$

Pattern – II

Equation (3) can be written as

$3v^2 - 3z^2 = z^2 - u^2$

$3(v + z)(v - z) = (z + u) z - u$

Four different choices of solution obtained are as follows:

Choice I

$X = 2A^2 - 6B^2 + 12AB$

$Y = 12B^2 - 4A^2$

$Z = A^2 + 3B^2$

A few interesting properties observed are as follows.

I) $x(1, B) + 12t_{3, B} \equiv 2 \pmod{18}$

II) $x(5, B) + 12t^{3, B} \equiv 16 \pmod{66}$

III) $x(10, B) + 12t_{3, B} \equiv 74 \pmod{126}$

IV) $x(A, 2) - 4t_{3, A} \equiv -2 \pmod{22}$

V) $x(A, 4) - 4t_{3, A} \equiv -4 \pmod{46}$

VI) $2x(A, A+1) + y(A, A+1) - 48t_{3, A} \equiv 0$

VII) $2x(A, A(A+1)) + y(A, A(A+1)) - 48P_A^5 \equiv 0$

VIII) $2x(A, (A+1)(A+2)) + y(A, (A+1)(A+2)) - 144P_3^A \equiv 0$

IX) Each of the following expression represents a nasty numbers

(a) $2x(A, B) + y(A, B)$

(b) $y(A, B) + 4z(A, B)$

(c) $y(3B, B)$

Choice II

$x = -4A^2 + 12B^2$

$y = 2A^2 - 6B^2 - 12AB$

$z = A^2 + 3B^2$

Choice III

$X = 2A^2 - 6B^2 + 12AB$

$Y = 12B^2 - 4A^2$

$Z = -A^2 - 3B^2$

Choice IV

$X = 2A^2 - 6B^2 - 12AB$

$Y = -4A^2 + 12B^2$

$z = -3B^2 - A^2$

Pattern – III

Equation (3) can be written as

$$u^2 + 3v^2 = 4z^2 * 1 \quad (7)$$

$$\text{Assume that } z = a^2 - 3b^2 \quad (8)$$

$$\text{Write 1 as } 1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \quad (9)$$

Use (8) and (9) in (7) and employing the method of factorization. Define

$$u + i\sqrt{3}\, v = \frac{(1 + i\sqrt{3})(1 + i\sqrt{3})}{2\,(10)} (a + i\sqrt{3}\, b)^2$$

Equating the real and imaginary parts in (10)

$$u = -a^2 + 3b^2 - 6ab \quad (11)$$

$$v = a^2 - 3b^2 - 2ab \quad (12)$$

Substituting (11) and (12) in (2), the corresponding integer solution of (1) are given by

$$\left.\begin{aligned} x &= 2a^2 - 6b^2 - 12ab \\ y &= -4a^2 + 12b^2 \\ z &= a^2 + 3b^2 \end{aligned}\right\} \quad (13)$$

A few interesting properties observed are as follows.

I) $x(1, b) + 12t_{3, b} \equiv 2 \pmod 6$

II) $x(2, b) + 12t_{3, b} \equiv 8 \pmod{18}$

III) $x(3, b) + 12t_{3, b} \equiv 18 \pmod{30}$

IV) $x(4, b) + 12t_{3, b} \equiv 22 \pmod{42}$

V) $x(a, 1) - 4t_{3, a} \equiv -6 \pmod{14}$

VI) $x(a, 2) - 4t_{3, a} \equiv -24 \pmod{26}$

VII) $x(a, 3) - 4t_{3, a} \equiv -16 \pmod{38}$

VIII) $x(a, 4) - 4t_{3, a} \equiv -46 \pmod{50}$

IX) Each of the following expression represents a nasty numbers

(a) $x(a, a) - 2y(a, a)$

(b) $y(a, b) + 4z(a, b)$

(c) $2x(a, b) + y(a, b)$

(d) $x(a, a) - 2z(a, a)$

Pattern – IV

Again, Equation (3) can be written as

$$u^2 + 3v^2 = 4z^2 * 1 \quad (14)$$

$$\text{Assume that } z = a^2 + 3b^2 \quad (15)$$

$$\text{Write 1 as } 1 = \frac{(1 + 4i\sqrt{3})(1 - 4i\sqrt{3})}{49} \quad (16)$$

Use (16) and (15) in (14) and employing the method of factorization. Define

$$u + i\sqrt{3}\, v = (1 + i\sqrt{3}) \frac{(1 + 4i\sqrt{3})}{7} (a + i\sqrt{3}\, b)^2 \quad (17)$$

Equating the real and imaginary parts in (17)

$$u = 1/7\,(33b^2 - 11a^2 - 30ab) \quad (18)$$

$$v = 1/7\,(5a^2 - 15b^2 - 22ab) \quad (19)$$

Our interest is to obtain the integer solutions, so that the values of u and v are integers for suitable choices of the parameters a and b.

put $a = 7A$, $b = 7B$

$$u = 231B^2 - 77A^2 - 210AB \quad (20)$$

$$v = 35A^2 - 154AB - 105B^2 \quad (21)$$

$$z = 49A^2 + 147B^2 \quad (22)$$

Substituting (20) and (21) in (2), the corresponding integer solutions of (1) are given by

$$\left.\begin{aligned} x &= 28A^2 - 84B^2 - 672AB \\ y &= 546B^2 - 182A^2 - 252AB \\ z &= 49A^2 + 147B^2 \end{aligned}\right\} \quad (23)$$

The equation (23) represents non-zero distinct integral solution of (1) on two parameters.

A few interesting properties observed are as follows.

I) $x(A,1) - 56t_{3,A} \equiv -84 \pmod{700}$

II) $x(A,3) - 56t_{3,A} \equiv -756 \pmod{2044}$

III) $x(A,5)-56t_{3,A} \equiv -2100 \pmod{3388}$
IV) $x(A,6)-56t_{3,A} \equiv -3024 \pmod{4032}$
V) $x(A,9)-56t_{3,A} \equiv -756 \pmod{6048}$
VI) $y(1,B)+504t_{3,B} \equiv -182 \pmod{798}$
VII) $y(2,B)+1008t_{3,B} \equiv -728 \pmod{1050}$
VIII) $y(4,B)+2016t_{3,B} \equiv -1358 \pmod{1554}$
IX) $y(7,B)+3528t_{3,B} \equiv -1988 \pmod{2310}$
X) $y(8,B)-1092t_{3,B} \equiv -1392 \pmod{2564}$

Pattern – V

Equation (3) may be equivalent to

$u^2+3v^2 = (2z)^2$ (24)

Which is satisfied by

$u=3p^2- q^2$ (25)

$v=2pq$ (26)

$z=1/2\ (3p^2+q^2)$ (27)

Our interest is to obtain the integer solutions, so that the values of z are integers for suitable choices of the parameters p and q.

put p =2 A, q =2 B in (25), (26) and (27) we get

$u=12A^2- 4B^2$ (28)

$v=8AB$ (29)

$z=6A^2+ 2B^2$ (30)

Substituting (28) and (29) in (2), the corresponding integer solutions of (1) are given by

$x=12A^2- 4B^2+24AB$

$y=12A^2- 4B^2- 24AB$

$z=6A^2+ 2B^2$

A few interesting properties observed are as follows.

(i) $x(A,4)-24t_{3,A} \equiv -64 \pmod{84}$
(ii) $x(A,7)-24t_{3,A} \equiv -40 \pmod{156}$
(iii) $y(1,B)+8t_{3,B} \equiv 12 \pmod{20}$
(iv) $y(3,B)+8t_{3,B} \equiv 40 \pmod{68}$
(v) $x(A,B)-y(A,B)-48AB \equiv 0$
(vi) $x(A,A+1)-y(A,A+1)-96t_{3,A} \equiv 0$
(vii) $x(A,A(A+1))-y(A,A(A+1))-96P_A^5 \equiv 0$
(viii) $x(A,(A+1)(A+2))-y(A,(A+1)(A+2))-288P_3^A \equiv 0$

4 Conclusion

In this paper we have presented five different patterns of non-zero distinct integer solutions of the ternary quadratic equation given by $x^2+ xy +y^2=12z^2$ To conclude, one may search for other patterns of solutions and their corresponding properties.

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