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Brief discussions on T-level complex fuzzy subgroup

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Abstract
In this article, we combine complex fuzzy set and fuzzy subgroups. We introduce the notion of T-level complex fuzzy subgroups, step complex fuzzy subgroups and study fundamental properties. Also the basic operations like union, intersection, product, Cartesian product of two complex fuzzy subgroups are studied. Furthermore, we define a T-level subset and T-level subgroups and then some of their properties are studied.

Keywords: Complex fuzzy group, T-level subgroup, T-norm. Step complex fuzzy group, Cartesian product, phase term

1. Introduction

The concept of the fuzzy set was first introduced by Zadeh in a seminal paper in 1965 [13]. This is the generalization of script Set in terms of membership function. The notion of the fuzzy set A on the universe of discourse U is the set of order pair $\{(x, \mu_A(x)), x \in U\}$ with a membership function $\mu_A(x)$, taking the value on the interval [0, 1]. In [10] Rosenfeld used this concept to develop the theory of fuzzy groups. In fact, many basic properties in group theory are found to be carried over to fuzzy groups. Anthony and Sherwood [1] redefined fuzzy subgroups in terms of a t-norm which replaced the minimum operation and they characterized basic properties of t-fuzzy subgroups in [1, 2]. Sherwood [11] defined products of fuzzy subgroups using t-norms and gave some properties of these products. Ramot. [9], extended the fuzzy set to complex fuzzy set with membership function $z = r_s e^{-iws(x)}$ where $i = \sqrt{-1}$, which ranges in the interval [0, 1] to a unit circle. Ramot *et al.* [8] also introduced different fuzzy complex operations and relations, like union, intersection, complement etc., still it is necessary to determine the membership functions correctly, which will give the appropriate or approximate result for real life applications. The membership function defined for the complex fuzzy set $z = r_s e^{iws(x)}$, which compromise an amplitude term $r_s(x)$ and phase term w_s. The amplitude term retains the idea of “fuzziness” and phase term signifies declaration of complex fuzzy set, for which the second dimension of membership is required. The complex fuzzy set allows extension of fuzzy logic that is i to continue with one dimension gradeness of membership. XinFu *et al.* [12] defined the fuzzy complex membership function of the form z=a+ib, where x, y are two fuzzy numbers with membership function $\mu_A(x)$, $\mu_B(x)$ respectively. If b does not exist, z degenerates to a fuzzy number.

Xin Fu *et al.* [12], also discussed a complex number in cartesian form where a= $r_s\cos(x)$ and b=$r_s\sin(x)$, which are in polar forms defined in [8]. The fuzzy number is created by interpolating complex number in the support of fuzzy set [3-5, 8].The member ship function are usually difficult to determine accurately and one may argue of accurate or precise membership function are necessary in reality.

Following the above resent development the fuzzy set theory Zhang *et al.* [14] studies d-equality of complex fuzzy set. This is the logical development since a complex membership function should more difficult that a real membership function to be determine in practice. Complex fuzzy set is a unique framework over the advantage of traditional fuzzy set. The support of complex fuzzy set is unrestricted, may include any kind of object such as number, name etc, which is off course a complex number. The notion of T-norm and T-co norm are used throughout this paper [6].

This paper is organized in the following order. The review of complex fuzzy set and operations on it are discussed in section 2 and section 3discussed t-level subgroups and its properties. The idea of step fuzzy complex subgroups are studied in section 4 followed by summary and suggestion for future work are given in section 5.

2. Complex fuzzy sets and operators

Definition -2.1: A complex fuzzy subset A, defined on a universe of discourse X, is characterized by a membership function τ_A(x) that assigns any element x∈X a complex valued grade of membership in A. The values of τ_A(x) all lie within the unit circle in the complex plane and thus all of the form $P_A(x)\, e^{j\mu_A(x)}$ where P_A(x) and $e^{j\mu_A(x)}$ are both real valued and P_A(x) ∈ [0, 1]. Here P_A(x) is termed as amplitude term and $e^{j\mu_A(x)}$ is termed as phase term.

The complex fuzzy set may be represented in the set form as A = {(x,τ_A(x)) / x∈X }. It is denoted by CFS.

The phase term of complex membership function belongs to (0,2π). Now we take those forms which Ramot. *et al.* presented in [8] to define the game of winner, neutral and lose.

$$\mu_{A\cup B}(x) = \begin{cases} \mu_A(x) & \text{if} p_A > p_B \\ \mu_B(x) & \text{if} p_A < p_B \end{cases}.$$

This is a novel concept and it is the generalization of the concept "winner take all" introduced by Ramot. *et al.* [8] for the union of phase terms.

Example 2.1: Let X = {x_1, x_2, x_3} be a universe of discourse. Let A and B be complex fuzzy sets in X as shown below.

A = {0.6 $e^{i(0.8)}$, 0.3 $e^{i\frac{3\pi}{4}}$,0.5 $e^{i(0.3)}$}

B = {0.8 $e^{i(0.9)}$, 0.1 $e^{i\frac{\pi}{4}}$,0.4 $e^{i(0.5)}$}

A∪B = {0.8 $e^{i(0.9)}$, 0.3 $e^{i\frac{3\pi}{4}}$,0.5 $e^{i(0.3)}$}

We can easily calculate the phase terms $e^{i\mu_{A\cap B}(x)}$ on the same line by winner, neutral and loser game.

Definition-2.2: Here a complex fuzzy subset A of a group G is said to be a complex fuzzy subgroup of G if for all x,y∈ G

(CFSG1) A(xy) ≥ min { A(x), A(y) },

(CFSG2) A(x^{-1}) ≥ A(x) where the product x and yis denoted by xy and the inverse of x by x^{-1}. It is well known and easy to see that a complex fuzzy subgroup G satisfies A(x)≤ A(e) and A(x^{-1}) = A(x) for all x ∈ G, where 'e' is the identity of G.

Definition-2.3 [11]: A triangular norm (briefly a t-norm) is a function T:[0,1] x [0,1] → [0,1] satisfying, for each p, q, r, s ∈ [0,1]

(1) T(p,1) = p
(2) T(p, q) ≤ T(r, s) if p≤r and q≤s,
(3) T(p, q) = T(q, p),
(4) T(p, T(q, r)) = T(T(p, q),r).

The following are the basic properties of operations on complex fuzzy sets.

Proposition2.1: Let A, B, C be those complex fuzzy sets on X. Then

(i) (A∪B) ∩ C = (A∩ C) ∪(B∩C)
(ii) (A∩B) ∪ C = (A∪ C) ∩(B∪C)

Proof: Let A, B, C be those complex fuzzy sets on X and μ_A (x),μ_B (x), μ_C (x) are membership functions. Then we have

$$\begin{aligned}
\tau_{(A\cup B)\cap C}(x) &= P_{(A\cup B)\cap C}(x)\, e^{i\mu_{(A\cup B)\cap C}(x)} \\
&= \min\{P_{A\cup B}(x), P_C(x)\} e^{i\min\{\mu_{A\cup B}(x),\mu_C(x)\}} \\
&= \min\{\max\{P_A(x),P_B(x)\}, P_C(x)\} e^{i\min\{\max\{\mu_A(x),\mu_B(x)\},\mu_C(x)\}} \\
&= \max\{\min\{P_A(x),P_C(x)\},\min\{P_B(x),P_C(x)\}\} e^{i\max\{\min\{\mu_A(x),\mu_C(x)\},\min\{\mu_B(x),\mu_C(x)\}\}} \\
&= \max\{P_{A\cap C}(x),P_{B\cap C}(x)\} e^{i\max\{\mu_{A\cap C}(x),\mu_{B\cap C}(x)\}} \\
&= P_{(A\cap C)\cup(B\cap C)}(x)\, e^{i\mu_{(A\cap C)\cup(B\cap C)}(x)} \\
&= \tau_{(A\cap C)\cup(B\cap C)}(x)
\end{aligned}$$

Similarly, we can show (ii).

Proposition 2.2: Let A and B be two complex fuzzy sets in X. Then

(i) (A∪B) ∩ A = A
(ii) (A∩B) ∪ A = A.

Proof: Let A and B be two complex fuzzy sets in X and μ_A (x),μ_B (x)are membership functions. Then we have

$$\begin{aligned}
\tau_{(A\cup B)\cap A}(x) &= P_{(A\cup B)\cap A}(x)\, e^{i\mu_{(A\cup B)\cap A}(x)} \\
&= \min\{P_{A\cup B}(x),P_A(x)\} e^{i\min\{\mu_{A\cup B}(x),\mu_A(x)\}} \\
&= \min\{\max\{P_A(x),P_B(x)\}, P_A(x)\} e^{i\min\{\max\{\mu_A(x),\mu_B(x)\},\mu_A(x)\}}
\end{aligned}$$

$= P_{\mathrm{A}}(\mathrm{x})\, e^{\mathrm{i}\mu_{\mathrm{A}}(\mathrm{x})}$

$= \tau_{\mathrm{A}}(\mathrm{x})$

Similarly, we can show (ii).

Proposition 2.3: Let A and B be two complex fuzzy sets in X. Then

(i) A ∨ (A ∧ B) = A

(ii) A ∧ (A ∨ B) = A(Absorption laws)

Proof

$\tau_{\mathrm{A}\cup(\mathrm{A}\cap\mathrm{B})}(\mathrm{x}) = P_{\mathrm{A}\cup(\mathrm{A}\cap\mathrm{B})}(\mathrm{x})\, e^{\mathrm{i}\mu_{\mathrm{A}\cup(\mathrm{A}\cap\mathrm{B})}(\mathrm{x})}$

$= \max\{P_{\mathrm{A}}(\mathrm{x}), P_{\mathrm{A}\cap\mathrm{B}}(\mathrm{x})\} e^{\mathrm{i}\max\{\mu_{\mathrm{A}}(\mathrm{x}),\mu_{\mathrm{A}\cap\mathrm{B}}(\mathrm{x})\}}$

$= \max\{P_{\mathrm{A}}(\mathrm{x}), \min\{P_{\mathrm{A}}(\mathrm{x}), P_{\mathrm{B}}(\mathrm{x})\}\ \} e^{\mathrm{i}\max\{\mu_{\mathrm{A}}(\mathrm{x}),\min\{\mu_{\mathrm{A}}(\mathrm{x}),\mu_{\mathrm{B}}(\mathrm{x})\}\}}$

$=\min\{\max\{P_{\mathrm{A}}(\mathrm{x}), P_{\mathrm{A}}(\mathrm{x})\},\quad \max\{P_{\mathrm{A}}(\mathrm{x}), P_{\mathrm{B}}(\mathrm{x})\}\ \} e^{\mathrm{i}\min\{\max\{\mu_{\mathrm{A}}(\mathrm{x}),\mu_{\mathrm{A}}(\mathrm{x})\},\max\{\mu_{\mathrm{A}}(\mathrm{x}),\mu_{\mathrm{B}}(\mathrm{x})\}\}}$

$= P_{\mathrm{A}}(\mathrm{x})\, e^{\mathrm{i}\mu_{\mathrm{A}}(\mathrm{x})}$

$= \tau_{\mathrm{A}}(\mathrm{x})$

Similarly, we can show (ii).

Proposition2.4: Let A and B be two complex fuzzy sets in X. Then

(i) A ∨ B = B ∨ A

(ii) A ∧ B = B ∧ A commutative Law

Proof: Let A and B be two complex fuzzy sets in X and $\mu_{\mathrm{A}}(\mathrm{x}), \mu_{\mathrm{B}}(\mathrm{x})$ are membership functions. Then we have

$\tau_{\mathrm{A}\cup\mathrm{B}}(\mathrm{x}) = P_{\mathrm{A}\cup\mathrm{B}}(\mathrm{x})\, e^{\mathrm{i}\mu_{\mathrm{A}\cup\mathrm{B}}(\mathrm{x})}$

$=\max\{P_{\mathrm{A}}(\mathrm{x}), P_{\mathrm{B}}(\mathrm{x})\} e^{\mathrm{i}\max\{\mu_{\mathrm{A}}(\mathrm{x}),\mu_{\mathrm{B}}(\mathrm{x})\}}$

$= \max\{P_{\mathrm{B}}(\mathrm{x}), P_{\mathrm{A}}(\mathrm{x})\} e^{\mathrm{i}\max\{\mu_{\mathrm{B}}(\mathrm{x}),\mu_{\mathrm{A}}(\mathrm{x})\}}$

$= P_{\mathrm{B}\cup\mathrm{A}}(\mathrm{x})\, e^{\mathrm{i}\mu_{\mathrm{B}\cup\mathrm{A}}(\mathrm{x})}$

$= \tau_{\mathrm{B}\cup\mathrm{A}}(\mathrm{x})$

Similarly, we can show (ii).

Definition-2.4: Let S be a groupoid and T is a t-norm. A function B: S→[0,1] is a subgroupoid of S iff for every x,yin S, B(xy) ≥ T{B(x),B(y)}. If S is group, a t-complex fuzzy subgroupoid B is a t- complex fuzzy subgroup of S iff for each x ∈ S, $\mathrm{B}(\mathrm{x}^{-1}) \geq$ B(x).

Definition-2.5: For each i = 1,2,….,n, let A_{i} be a t- complex fuzzy subgroup in a group X_{i}. Let T be a T-norm. The T-product of A_{i} (i = 1,2,….,n) is the function,

$\mathrm{A}_1 \times \mathrm{A}_2 \times \dots \times \mathrm{A}_{\mathrm{n}} : \mathrm{X}_1 \times \mathrm{X}_2 \times \dots \times \mathrm{X}_{\mathrm{n}} \to [0,1]$ defined by

$(\mathrm{A}_1 \times \mathrm{A}_2 \times \dots \times \mathrm{A}_{\mathrm{n}})(\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_{\mathrm{n}}) = \mathrm{T}(\mathrm{A}_1(\mathrm{x}_1), \mathrm{A}_2(\mathrm{x}_2), \dots, \mathrm{A}_{\mathrm{n}}(\mathrm{x}_{\mathrm{n}}))$.

Definition-2.6: For each i = 1,2,….,n, let A_{i} be a complex fuzzy subgroup under a minimum operation in a group X_{i}, the membership function of the product

$\mathrm{A} = \mathrm{A}_1 \times \mathrm{A}_2 \times \dots \times \mathrm{A}_{\mathrm{n}}$ in $\mathrm{X} = \mathrm{X}_1 \times \mathrm{X}_2 \times \dots \times \mathrm{X}_{\mathrm{n}}$ is defined by

$(\mathrm{A}_1 \times \mathrm{A}_2 \times \dots \times \mathrm{A}_{\mathrm{n}})(\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_{\mathrm{n}}) = \min\{(\mathrm{A}_1(\mathrm{x}_1), \mathrm{A}_2(\mathrm{x}_2), \dots, \mathrm{A}_{\mathrm{n}}(\mathrm{x}_{\mathrm{n}})\}$.

Definition-2.7: Let A be a complex fuzzy subset of a set X and let t ∈ [0,1]. The set

A_t ={ x∈ X : A(x) ≥ t} is called a level subset of A.

Definition-2.8: A complex fuzzy subgroup A of a group X is called complex fuzzy normal if for all x,y in G it fulfills the following condition: A(xy)= A(yx).

Definition-2.9: A complex fuzzy subgroup A of a group X is said to be conjugate to a complex fuzzy subgroup B of G if there exists x in G such that for all g in G A(g) = $\mathrm{B}(\mathrm{x}^{-1}\mathrm{gx})$.

The following are the properties of Complex fuzzy sets

Theorem-2.1: Every complex fuzzy subset A of the universe of discourse X is a complex fuzzy subgroup of X.

Proof: Let A be a complex fuzzy subset of X. Then for all x,y ∈ X.

CFG-1 $\mathrm{A}(\mathrm{xy}) = \mathrm{p}(\mathrm{xy})\, \mathrm{e}^{\mathrm{j}\mu(\mathrm{xy})}$

$\geq \min\{\mathrm{p}(\mathrm{x}).\mathrm{p}(\mathrm{y})\} .\mathrm{e}^{\mathrm{j}\min\{\mu(\mathrm{x}) \wedge \mu(\mathrm{y})\}}$

$\geq \min\{\mathrm{p}(\mathrm{x}).\mathrm{p}(\mathrm{y})\}.\mathrm{e}^{\mathrm{j}\mu(\mathrm{x})}.\mathrm{e}^{\mathrm{j}\mu(\mathrm{y})}$

$\geq \min\{\mathrm{p}(\mathrm{x})\, \mathrm{e}^{\mathrm{j}\mu(\mathrm{x})}, \mathrm{p}(\mathrm{y}) \mathrm{e}^{\mathrm{j}\mu(\mathrm{y})}\}$

$\geq \min\{\mathrm{A}(\mathrm{x}), \mathrm{A}(\mathrm{y})\}$

CFG-2 $\mathrm{A}(\mathrm{x}^{-1}) = \mathrm{p}(\mathrm{x}^{-1})\, \mathrm{e}^{\mathrm{j}\mu(\mathrm{x}^{-1})} \geq \mathrm{p}(\mathrm{x})\, \mathrm{e}^{\mathrm{j}\mu(\mathrm{x}^{-1})} \geq \mathrm{p}(\mathrm{x})\, \mathrm{e}^{\mathrm{j}\mu(\mathrm{x})} \geq \mathrm{A}(\mathrm{x})$

Complex fuzzy subgroup conditions satisfied.

Therefore A forms a Complex fuzzy subgroup of X.

Theorem 2.2: Intersection of non-empty collection of complex fuzzy subgroup is a complex fuzzy subgroup.

Proof:

CFG-1 $(\bigcap_{i\in I} A_i)$ (xy) = $\mathrm{Inf}_{i\in I} A_i(xy)$

$\geq \mathrm{Inf}_{i\in I}\{\min\{A_i(x), A_i(y)\}$

$\geq \min\{\mathrm{Inf}_{i\in I} A_i(x), \mathrm{Inf}_{i\in I} A_i(y)\}$

$\geq \min\{\bigcap_{i\in I} A_i\ (x), \bigcap_{i\in I} A_i\ (y)\}$

CFG-2 $(\bigcap_{i\in I} A_i)\ (x^{-1}) = \mathrm{Inf}_{i\in I} A_i(x^{-1}) \geq \mathrm{Inf}_{i\in I} A_i(x^{-1}) \geq \bigcap_{i\in I} A_i(x)$

Definition-2.10: Let A and B be two complex fuzzy sets on X, and $\tau_A(x) = r_A(x)\ e^{j\mu_A(x)}$ and $\tau_B(x) = r_B(x)\ e^{j\mu_B(x)}$ be their membership functions, respectively. The union and intersection
of A and B if denoted as A∪B and A∩B, which is specified by a function

$\tau_{A\cup B}(x) = r_{A\cup B}(x)\ e^{j\mu_{A\cup B}(x)} = (r_A(x) \vee r_B(x))\ e^{j(\mu_A(x)\vee\mu_B(x))}$, where ∨ denote the max operator.

$\tau_{A\cap B}(x) = r_{A\cap B}(x)\ e^{j\mu_{A\cap B}(x)} = (r_A(x) \wedge r_B(x))\ e^{j(\mu_A(x)\wedge\mu_B(x))}$, where ∧ denote the min operator.

Theorem-2.3: Union of two complex fuzzy subgroups is also a complex fuzzy subgroup.
Proof: Let A and B be two complex fuzzy subgroups.
Now let x,y ∈ X,

(CFG-1) $\tau_{A\cup B}(xy) = r_{A\cup B}(xy)\ e^{j\mu_{A\cup B}(xy)}$

$= (r_A(xy) \vee r_B(xy))\ e^{j(\mu_A(xy)\vee\mu_B(xy))}$

$\geq (\ (r_A(x) \wedge r_A(y)\ )\vee (\ (r_B(x) \wedge r_B(y)\ )\ e^{j((\mu_A(x)\wedge\mu_A(y))\vee(\mu_B(x)\wedge\mu_B(y)))}$

$\geq (\ (r_A(x) \vee r_A(y)\ )\ \wedge(\ (r_B(x) \vee r_B(y)\ )\ e^{j((\mu_A(x)\vee\mu_A(y))\wedge(\mu_B(x)\vee\mu_B(y)))}$

$\geq (r_A(x) \vee r_B(x)\ )\ e^{j\mu_{A\cup B}(x)} \wedge (r_A(y) \vee r_B(y)\ e^{j\mu_{A\cup B}(y)}$

$\geq \min\{\tau_{A\cup B}(x), \tau_{A\cup B}(y)\}$

(CFG-2) $\tau_{A\cup B}(x^{-1}) = r_{A\cup B}(x^{-1})\ e^{j\mu_{A\cup B}(x^{-1})}$

$= (r_A(x^{-1}) \vee r_B(x^{-1}))\ e^{j(\mu_A(x^{-1})\vee\mu_B(x^{-1}))}$

$\geq (r_A(x) \vee r_B(x))\ e^{j(\mu_A(x)\vee\mu_B(x))} = r_{A\cup B}(x)\ e^{j\mu_{A\cup B}(x)} \geq \tau_{A\cup B}(x)$

Theorem 2.4: Intersection of two complex fuzzy subgroups is also a complex fuzzy subgroup.

Proof: The result is obvious.

Theorem 2.5: The product of complex fuzzy subgroup is also a complex fuzzy subgroup.
Proof: Let A and B be two complex fuzzy subgroups of X.
Now let x,y ∈ X,

(CFG-1) $\tau_{A\cup B}(xy) = r_{A\cup B}(xy)\ e^{j\mu_{A\cup B}(xy)}$

$= (r_A(xy)\ .\ r_B(xy))\ e^{j2\pi\left(\frac{\mu_A(xy)}{2\pi}\ .\ \frac{\mu_B(xy)}{2\pi}\right)}$

$\geq(\ \min\{r_A(x), r_A(y)\}.\ \min\{r_B(x), r_B(y)\})\ e^{j2\pi\left(\frac{\min\{\mu_A(x),\mu_A(y)\}}{2\pi}, \frac{\min\{\mu_B(x),\mu_B(y)\}}{2\pi}\right)}$

$\geq \min\{r_A(x).r_B(x),\ r_A(y).r_B(y)\} e^{j2\pi\left(\frac{\mu_{A\cup B}(x)}{2\pi}\ .\ \frac{\mu_{A\cup B}(y)}{2\pi}\right)}$

$\geq \min\{r_{A\cup B}(x), r_{A\cup B}(y)\ \} e^{j\mu_{A\cup B}(x)}\ .\ e^{j\mu_{A\cup B}(y)}$

$\geq \min\{r_{A\cup B}(x)\ e^{j\mu_{A\cup B}(x)}, r_{A\cup B}(y)\ e^{j\mu_{A\cup B}(y)}\}$

$\geq \min\{\tau_{A\cup B}(x), \tau_{A\cup B}(y)\}$

CFG-1 is satisfied.

(CFG-2) $\tau_{A\cup B}(x^{-1}) = r_{A\cup B}(x^{-1})\ e^{j\mu_{A\cup B}(x^{-1})} \geq r_{A\cup B}(x)\ e^{j\mu_{A\cup B}(x)} \geq \tau_{A\cup B}(x)$CFG-2 is satisfied.

Definition-2.11: Let A and B be two complex fuzzy sets on X, and $\tau_A(x) = r_A(x)\ e^{j\mu_A(x)}$ and $\tau_B(x) = r_B(x)\ e^{j\mu_B(x)}$ be their membership functions, respectively. The complex product ofA and B, denoted as AoB and is specified by a function

$\tau_{AoB}(x) = r_{AoB}(x)\ e^{j\mu_{AoB}(x)} = (r_A(x)\ .\ r_B(x))\ e^{j2\pi\left(\frac{\mu_A(x)}{2\pi}\ .\ \frac{\mu_B(x)}{2\pi}\right)}$

Definition 2.12: Let A and B be two complex fuzzy sets on X. Then the cartesian product ofA and B is defined as

(AxB) (x,y) = $r_{AxB}(x,y)\ e^{j\mu_{AxB}(x,y)}$

$= \min\{r_A(x), r_B(y)\}.\ e^{j(\mu_A(x)\vee\mu_B(y))}$

$= \min\{r_A(x), r_B(y)\}.\ e^{j\mu_A(x)}.\ e^{j\mu_B(y)}$

$= \min\{r_A(x)e^{j\mu_A(x)}, r_B(y)\ e^{j\mu_B(y)}\}$

= min {A(x),B(x)} where $A(x) = r_A(x)\ e^{j\mu_A(x)}$

Note 2.6: Complement of complex fuzzy subgroup is not a complex fuzzy subgroup.

Proof: By definition,

$\tau_{C(S)}(x) = P_{C(S)}(x)\ e^{j\mu_{C(S)}(x)}$

$= (1\text{-}P_S(x))\ e^{j(2\pi-\mu_S(x))}$

Now, $\tau_{C(S)}(xy) = (1-P_S(xy))\, e^{j(2\pi-\mu_S(xy))} = 1 - \tau_S(xy)$

$\leq 1-\min\{\tau_S(x), \tau_S(y)\}$

$\leq \max\{1-\tau_S(x), 1-\tau_S(y)\}$

$\leq \max\{\tau_{C(S)}(x), \tau_{C(S)}(y)\}$

Complement of complex fuzzy subgroup is not a complex fuzzy subgroup.

Proposition 2.5: For all a,b∈ [0,1] and 'm' is any positive integer, verify that

(i) If $a < b$, then $a^m < b^m$ and

(ii) $\text{Min}\{a,b\} = \min\{a^m,b^m\}$.

Proof: It is obvious.

Proposition2.6: If X is a group, then $X^m = \{x, (A_X(x))^m / x \in X\}$ is a complex fuzzy subgroup.

Proof: Let X is a complex fuzzy subgroup, where (X,) is a group. Thus $(X^m,.)$ is a group for all positive integer m. Let 'm' is a positive integer and $x,y \in X^m$.

(CFG-1) $(A\, m(xy)) = (A\,(xy))^m$

$\geq \min\{A(x), A(y)\}^m$

$= \min\{(A\,(x))^m,(A\,(y))^m\}$

$= \min\{mA(x), mA(y)\}$

(CFG-1) $(A\, m(x^{-1})) = (A\,(x^{-1}))^m \geq (A\,(x))^m \geq (A\, m(x))$

$\therefore (X^m, .)$ is a complex fuzzy subgroup.

Corollary-2.7: The complex fuzzy subgroup X^n is a complex fuzzy subgroup of X^m, if $m \leq n$.

Proof: Clearly X^n and X^m are complex fuzzy subgroups by above theorem, for all x ∈ [0,1],

$X^m \geq X^n$ implies then $X^n \subset X^m$. (Since $An(x) \leq Am(x)$ for all x ∈ X).

3. T-level subgroup

In this section, we introduce a definition of a t-level subset of a complex fuzzy subset and then we give some of the important algebraic results.

Definition-3.1: Let 'A' be a complex fuzzy subset of a set X, T a t-norm and r ∈[0,1]. Then we define a T-level subset of a complex fuzzy subset A as

$A_r^T = \{x \in G / T(A(x),r) \geq r\}$.

The following are the main results based on the above definition 3.1

Theorem-3.1: Let X be a group and A be T-complex fuzzy subgroup of X, then the t-level subset A_r^T, r ∈ [0,1], is a subgroup of X.

Proof: $A_r^T = \{x \in G / T(A(x),r) \geq r\}$ is clearly non-empty.

Let $x, y \in A_r^T$, then $T(A(x),r) \geq r$ and $T(A(y),r) \geq r$.

Since 'A' is T-complex fuzzy subgroup of X, $A(xy) = p(xy)\, e^{j\mu(xy)} \geq T(A(x), A(y))$ is satisfied.

This means $T(A(xy), r) \geq T(T(A(x).A(y)),r) = T(A(x), T(A(y),r)) \geq T(A(x),r) \geq r$.

Hence $xy \in A_r^T$.

Since 'A' is T-complex fuzzy subgroup, $A(x^{-1}) = A(x)$ and hence $T(A(x^{-1}), r) = T(A(x), r) \geq r$.

This means that $x^{-1} \in A_r^T$. Therefore A_r^T is a subgroup of X.

Theorem-3.2: Let X be a group and A be a complex fuzzy subgroup of X, then the T-level subset A_r^T, r ∈ [0,1], $T(A(e),r) \geq r$, is a subgroup of X, where 'e' is the identity of X.

Proof: $A_r^T = \{x \in G/T(A(x),r) \geq r\}$ is clearly non-empty.

Let $x, y \in A_r^T$, then $T(A(x),r) \geq r$ and $T(A(y),r) \geq r$.

Since 'A' is a subgroup of X, $A(xy) = p(xy)\, e^{j\mu(xy)} \geq \min\{A(x), A(y)\}$ is satisfied.

This means that $T(A(xy), r) \geq T(\min\{A(x),A(y)\},r)$, where there are two cases:

$\min\{A(x), A(y)\} = A(x)$ or $\min\{A(x), A(y)\} = A(y)$.

Since $x, y \in A_r^T$, also in two cases $T((\min\{A(x), A(y)\}),r) \geq r$.

Therefore $T(A(xy),r) \geq r$. Thus we get $xy \in A_r^T$. It is easily seen that, as above $x^{-1} \in A_r^T$. Hence A_r^T is a subgroup of X.

Theorem-3.3: Let X be a group and A be a complex fuzzy subset of X such that A_r^T is a subgroup of X for all r ∈ [0,1], $T(A(x),r) \geq r$, then A is a T-fuzzy complex subgroup of X.

Proof: Let x, y ∈ X and let $T(A(x),r_1) = r_1$ and $T(A(y),r_2) = r_2$.

Then $x \in A_{r_1}^T$, $y \in A_{r_2}^T$. Let us assume that $r_1 < r_2$. Then these follows $T(A(x),r_1) < T(A(y),r_2)$

And $A_{r_2}^T \subseteq A_{r_1}^T$. So $y \in A_{r_1}^T$. Thus $x, y \in A_{r_1}^T$ and since $A_{r_1}^T$ is a subgroup of X, by hypothesis,

$xy \in A^T_{r_1}$. Therefore $T(A(xy),r_1) \geq r_1 = T(A(x),r_1)$, $T(A(y),r_1) = T(T(A(x),A(y),r_1)$.
Thus we get $T(A(xy),r_1) \geq T(T(A(x),A(y),r_1)$. As a T-norm is monotone with respect to each variable and symmetric, we have $A(xy) = p(xy)\, e^{j\mu(xy)} \geq T(A(x),A(y))$.
Next, let $x \in X$ and $T(A(x),r)= r$. Then $x \in A^T_r$. Since A^T_r is a subgroup, $x^{-1} \in A^T_r$.
Therefore $T(A(x^{-1}),r) \geq r$ and hence $T(A(x^{-1}),r) \geq T(A(x),r)$. So we have $A(x^{-1}) \geq A(x)$.
Thus A is a T-complex fuzzy subgroup of X.

Theorem-3.4: Let A and B be t-level subsets of the sets X_1 and X_2, respectively, and let
$r \in [0,1]$. Then AxB is also a T-level subset of $X_1 x X_2$.
Proof: Since any T-norm T is associative, using definition-2.5 and definition-3.1, we can write the following statements,
$T((AxB)(a,b),r) = T(T(A(a),B(b)),r) = T(A(a),T(B(b),r) \geq T(A(a), r) \geq r$.
This completes the proof.

Definition-3.2: Let X be a group and A be a T-complex fuzzy subgroup of X. The subgroups A^T_r,
$r \in [0,1]$ and $T(A(x),r) \geq r$ are called T-level subgroup of A.
The following theorems are proved based on the definition 3.2

Theorem-3.5: Let X_1 and X_2 be two groups. A and B a T-complex fuzzy subgroup of X_1 and X_2 respectively. Then the T-level subset$(AxB)^T_r$; for $r \in [0,1]$ is a subgroup of $X_1 x X_2$, where e_{X_1}
and e_{X_2} are identities of X_1 and X_2 respectively.

Proof: $(A x B)^T_r = \{(x,y) \in X_1 x X_2 ; T((AxB)(x,y), r) \geq r\}$.
Since $T((A x B)(e_{X_1}, e_{X_2}), r) = T(T(A(e_{X_1}),B(e_{X_2}),r) \geq T(A(e_{X_1}),r) \geq r$
$\therefore$ $(A x B)^T_r$ is non empty. Let (x_1,y_1), $(x_2,y_2) \in (A x B)^T_r$, then $T((A x B)(x_1,y_1), r) \geq r$ and
$T((A x B)(x_2,y_2), r) \geq r$. Since A x B is a t-complex fuzzy subgroup of $X_1 x X_2$,
we get $(A x B)((x_1,y_1)\ (x_2,y_2)) = T(A(x_1x_2), B(y_1y_2))$.
Using A and B are t-complex fuzzy subgroup, we get
$T\big((A x B)(x_1x_2, y_1y_2)\big) \geq T(T(A(x_1x_2), B(y_1y_2),r)$
$= T(A(x_1x_2),T(B(y_1y_2),r)) \geq T(A(x_1x_2),r) \geq r$
Hence (x_1,y_1), $(x_2,y_2) \in (A x B)^T_r$. Again $(x,y) \in (AxB)^T_r$ implies
$T((A x B)(x,y)^{-1}, r) = T((A x B)(x^{-1}, y^{-1}), r)$
$= T(T(A(x^{-1}), B(y^{-1})),r) = T(A(x^{-1}), T(B(y^{-1})),r) \geq T(A(x^{-1}),r) \geq r$
This means that $(x,y)^{-1} \in (A x B)^T_r$. Therefore $(A x B)^T_r$ is a subgroup of $X_1 x X_2$.

Theorem-3.6: Let X be a group and A^T_r a T-level subgroup of X. If A is a normal t-complex fuzzy subgroup, then A^T_r is a normal subgroup of X.

Proof: By theorem-3.1 A^T_r is a T-level subgroup of X. Now let us show that A^T_r is normal. For all
$a \in X$ and $x \in A^T_r$, $T(A(axa^{-1}),r) = T(A(aa^{-1}x),r) = T(A(x),r) \geq r$.
Thus $axa^{-1} \in A^T_r$. Hence A^T_r is a normal subgroup of X.

Theorem-3.7: Let A,B be complex fuzzy subsets of the sets X_1 and X_2, respectively, T be a
t-norm and $r \in [0,1]$. Then A^T_r x $B^T_r = (AxB)^T_r$.

Proof: Let (a,b) be an element of A^T_r x B^T_r. Then $a \in A^T_r$ and $b \in B^T_r$. By definition-3.1, we can write $T(A(a),r) \geq r$ and $T(B(b),r) \geq r$.
Using definition-2.3 and definition-2.5 we get
$T((A x B)(a,b), r) = T(T(A(a), B(b)),r) = T(A(a), T(B(b)),r) \geq T(A(a),r) \geq r$.
Thus we have $(a,b) \in (AxB)^T_r$. Now let $(a,b) \in (AxB)^T_r$, this is required following statements
$T((A x B)(a,b), r) = T(T(A(a), B(b)),r) = T(A(a), T(B(b)),r) \geq r = T(1,r)$. Thus the inequality
$T(A(a),r) \geq r$ and $T(B(b),r) \geq r$ is satisfied. Hence (a,b) is in A^T_r x B^T_r.
This completes the proof.

Theorem-3.8: Let $A_1, A_2, \dots, A_n$ be complex fuzzy subgroups under a minimum operation in groups $X_1, X_2, \dots, X_n$ respectively, $r \in [0,1]$. Then
$(A_1 x A_2 x \dots x A_n)^T_r = A^T_{1r} x A^T_{2r} x \dots x A^T_{nr}$.

Proof: Let $(a_1, a_2, \dots, a_n)$ be an element of $(A_1 x A_2 x \dots x A_n)^T_r$.
Using definition-2.3 and definition-2.5 we can write
$T(\min((A_1 x A_2 x \dots x A_n)(a_1, a_2, \dots, a_n),r) = T(\min(A_1(a_1), A_2(a_2), \dots, A_n(a_n)),r)$.
For all i=1,2,…,n, $\min(A_1(a_1), A_2(a_2), \dots, A_n(a_n)) = A_i(a_i)$.
This gives us $T(\min(A_1(a_1), A_2(a_2), \dots, A_i(a_i), \dots A_n(a_n)),r) = T(A_i(a_i),r) \geq r$.
Thus we have $a_i \in A^T_{ir}$. That is $(a_1, a_2, \dots, a_n) \in A^T_{1r} x A^T_{2r} x \dots x A^T_{nr}$.
Similarly, let $(a_1, a_2, \dots, a_n)$ be an element of $A^T_{1r} x A^T_{2r} x \dots x A^T_{nr}$. Then for all i=1,2,…,n,

$a_i \in A^T_{ir}$. That is, T($A_i(a_i)$,r) ≥r.
Since $\min(A_1(a_1), A_2(a_2), \ldots, A_i(a_i), \ldots A_n(a_n))$=$A_i(a_i)$ and T($A_i(a_i)$,r) ≥r, we get
T((A_1 x A_2 x …. x A_n) ($a_1, a_2, \ldots, a_n$),r) = T($\min(A_1(a_1), A_2(a_2), \ldots, A_i(a_i), \ldots A_n(a_n))$,r)
= T($A_i(a_i)$,r) ≥r.
Thus $(a_1, a_2, \ldots, a_n) \in (A_1$ x A_2 x …. x $A_n)\,^T_r$.
Finally, this completes the proof.

4. Step complex fuzzy subgroup

Let X be a group and T be a t-norm. We have the following.

Definition-4.1: Let 'A' be a complex fuzzy subgroup of X with respect to T. Then 'A' is called step complex fuzzy subgroup of X with respect to T if for every$\lambda \in$ [0,1] A_λ is a subgroup of X
when $A_\lambda \neq \emptyset$.

Theorem-4.1: Let 'A' be a complex fuzzy subgroup of X with respect to T. Then λ isa step complex fuzzy subgroup of X with respect to T if and only if A(xy) ≥ min{ A(x),A(y) }
for all x,y∈ X.

Proof: Suppose A(xy) ≥ min{ A(x),A(y) } for all x,y ∈ X.
When $A_\lambda \neq \emptyset$, $\lambda \in$ [0,1]. Let x,y∈ A_λ. Then
A(xy^{-1}) ≥ min{ A(x),A(y^{-1}) } = min{ A(x),A(y) } ≥ λ.
Hence $xy^{-1} \in A_\lambda$is a subgroup of X.
Conversely, suppose there are x,y∈ X such that A(xy) ≥ min{ A(x),A(y) }.
Put λ= min{ A(x),A(y) }, then x,y∈ A_λ.Because A(xy) ≤ min{ A(x),A(y) }= λ.
∴xy ∉ A_λ. $A_\lambda$is not a subgroup of X.
Consider the t-norm T, T_0(a,b) = min{a,b}, for all (a,b) ∈ [0,1]x[0,1]. We get the following corollary.

Corollary-4.2: Suppose A is a complex fuzzy subgroup of X with respect to T. Then
(i) A is a step complex fuzzy subgroup of X with respect to T_0.
(ii) Furthermore, A is a step complex fuzzy subgroup of X with respect to T, where T is a arbitrary t-norm.

Proof: (i) Because A(xy) ≥ T_0(A(x),A(y)) = min{ A(x),A(y) } for all x,y ∈ X, it follows from the above theorem that A is a step complex fuzzy subgroup of X with respect to T_0.
(ii) This conclusion follows from (i) and that fact T_0 ≥ T.

Definition-4.2: Let 'A' be a step complex fuzzy subgroup of X with respect to T. Suppose A(X) is the range of A. A(X) = { A(x) / x ∈ X }.
If the A(X) is finite; A(X) = { a_0,a_1,….,a_m }, with 1≥ a_0 ≥ a_1 ≥….≥ a_m ≥0, the integer m is called the length of A.
If the A(X) is countable; A(X) = { a_0,a_1,….,a_m,… }, with
1≥ a_0 ≥ a_1 ≥….≥ a_m ≥….≥ 0, the length of A is defined by +∞.

Theorem-4.3: Let 'A' be a step complex fuzzy subgroup of X with respect to T, the length of A is ℓ. A(X) = { a_0,a_1,….,$\frac{a}{2}$ }, with 1≥ a_0 ≥ a_1 ≥….≥ $\frac{a}{2}$ ≥ 0. Suppose
A= {A_λ / $\lambda \in$ [0,1],$A_\lambda \neq \emptyset$ }. Then B = {A_{C_1},A_{C_2},….,A_{C_ℓ}} and we can get a subgroup series without repetition in X: {e} ≤ A_{a_0} ≤ A_{a_1} ≤….≤ A_{a_ℓ} = X.

Proof: Let i be a integer.
Case-(i) : If 0≤ i ≤ℓ, then there exists a $x_i \in$ X; A(x_i) = a_i, hence $A_{a_i} \neq \emptyset$. By definition 3.1,
A_{a_i} (i=0,1,2,…, ℓ) is a subgroup of X.
Case-(ii) : If 1≤ i <i+1 ≤ℓ, then clearly $A_{a_i} \subseteq A_{a_{i+1}}$. Because there is a $x_{i+1} \in$ X such that
A(x_{i+1}) =a_{i+1}. And a_{i+1}<a_i, then $x_{i+1} \in A_{a_{i+1}}$, $x_{i+1} \notin A_{a_i}$. Hence $A_{a_i} \neq A_{a_{i+1}}$,
A_{a_i}<$A_{a_{i+1}}$.
Because e = min A(X), then for all x ∈ X, A(x) ≥ a_ℓ, thus $A_{i_\ell} \supseteq$ X. Hence A_{a_ℓ} = X.
Therefore, we get sequence {e} ≤ A_{a_0} ≤ A_{a_1} ≤….≤ A_{a_ℓ} = X.
For all $A_\lambda \in$ B, if $a_{i+1} \leq \lambda \leq a_i$, 1≤i+1 ≤$\ell$, then
A_λ= {x / x ∈ X, A(x)≥ λ} = {x / x ∈ X, A(x)≥ a_i} ∪ {x / x ∈ X, λ≤ A(x) ≤ A_λ}=A_{a_i} ∪ ∅= A_{a_i}.
If λ>a_0, then A_λ=∅, $A_\lambda \notin$ B.
If λ<a_ℓ, then $A_\lambda \supseteq A_{a_\ell}$ = X, hence A_λ=A_{a_ℓ}.
∴ B⊆{A_{a_0},A_{a_1},A_{a_2}, …. , A_{a_ℓ}}.

5. Summary and future work: The work presented in this paper is the novel frame work of step complex fuzzy subgroups. In this paper the various properties and operation of complex fuzzy set are investigated. We also presented the properties of t-level complex fuzzy subgroups. A further study is required to implement these notions in real life applications such as soft set, Bipolar fuzzy complex set, vague set and rough set etc.

6. References

1. Anthony JM, Sherwood H. Fuzzy groups redefined, J. Math. Anal. Appl. 1979; 69:124-130.
2. Anthony JM, Sherwood H. A characterization of fuzzy subgroups, Fuzzy Sets and Systems. 1982; 7:297-305.
3. Buckley JJ. Fuzzy Complex Number in Proceedings of ISFK, Gaungzhou, China, 1987, 597-700.
4. Buckley JJ. Fuzzy Complex Number", Fuzzy Sets &Systems.1989; 33:333-345.
5. Buckley JJ. Fuzzy Complex Analysis I: Differentiation, Fuzzy Sets &Systems. 1991; 4:269-284.
6. Jang JSR, Sun CT, Mizutani E. Neuro-Fuzzy and Soft Computing, PHI, New Delhi, 2005.
7. Pedrycz, Gomide F. Introduction to Fuzzy Set Analysis and Design, PHI, New Delhi, 2004.
8. Ramot Daniel, Menahem Friedman, Gideon Langholz, Abraham Kandel. Complex Fuzzy Logic, IEEE Transactions on Fuzzy Systems, 2003, 11-4.
9. Ramot Daniel, Menahem Friedman, Gideon Langholz, Abraham Kandel, Complex Fuzzy Set, IEEE Transactions on Fuzzy Systems, 2002, 10-2.
10. Rosenfeld A, Fuzzy groups J. Math Anal Appl. 1971; 35:51-517.
11. Sherwood H. Products of fuzzy subgroups, Fuzzy Sets and Systems. 1983; 11:79-89.
12. Xin Fu, Qiang Shen. A Novel Framework of Complex Fuzzy Number and Its Applicationto Computational Modeling, FUZZ –IEEE 2009, Korea, 2009, 20-24.
13. Zadeh LA. Fuzzy Sets, Information & Control.1965; 8:338-353.
14. Zhang Guangquan, Tharam Singh Dillon, Kai-yung Cai Jun Ma, Jie Lu. δ-Equality of Complex Fuzzy Relations, 24th IEEE International Conference on Advance Information Networking and Application, 2010.
15. Zhang GQ. Fuzzy distance and limit of fuzzy number, fuzzy system and mathematics. 1992; 6(1):21-28.
16. Zhang GQ Fuzzy limit theory of fuzzy complex number, fuzzy Set system.1992; 46:227-235.

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