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Comparative study of vector space and modules

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Abstract
A vector space also called linear space is a collection of objects called vectors, which may be added together and multiplied by numbers, called scalars in this context. Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by complex numbers, rational numbers, or generally any field. And a module is one of the fundamental algebraic structures used in abstract algebra. A module over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of an arbitrary given ring and a multiplication is defined between elements of the ring and elements of the module. Both seem to be same in terms of definition but when analysed deeply they are quite different as discussed in this paper.

Keywords: Vector space, modules, scalars, subspace, sub-module

Introduction

Let F be a field. A set V is called a vector space over F if there is an operation of addition

$$(x,y) \rightarrow x+y$$

On V, and a scalar multiplication function

$$(\alpha, x) \rightarrow \alpha x$$

From F×V to V, such that the following properties are satisfied.

(i) $(u+v) + w = u + (v+w)$ for all u, v, w ∈ V
(ii) $u + v = v + u$ for all u, v ∈ V
(iii)There exist an element 0 ∈ V such that $0 + v = v + 0$ for all v ∈ V
(iv) For each v ∈ V there exist u ∈ V such that $u + v = 0$.
(v) $1v = v$ For all v ∈ V
(vi) $(\alpha\beta)v = \alpha(\beta v)$ for all $\alpha, \beta \in F$ and $v \in V$.
(vii) $(\alpha + \beta)v = \alpha v + \beta v$, for all $\alpha, \beta \in F$ and$v \in V$.
(viii) $\alpha(u + v) = \alpha u + \alpha v$, for all $\alpha \in F$ and$v, u \in V$.

The field of scalars F and the vector space V both have zero elements which are both commonly denoted by the symbol '0'. With a little care one can always tell from the context whether 0 means the zero scalar or the zero vector. Sometimes we will distinguish notationally between vectors and scalars by writing a tilde underneath vectors; thus 0 would denote the zero of the vector space under discussion.

Example. Let P be the Euclidean plane and choose a fixed point O∈ P. The set of all line segments OP, as P varies over all points of the plane, can be made into a vector space over R, called the space of position vectors relative to O. The sum of two line segments is defined by the parallelogram rule; that is, for P, Q∈ P find the point R such that OPRQ is a parallelogram, and define OP+OQ=OR. If λ is a positive real number then λOP is the line segment OS such that S lies on OP or OP produced and the length of OS is λ times the length of OP. For negative λ the product λOP is defined similarly (giving a point on PO produced).

Subspace

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

In general, all ten vector space axioms must be verified to show that a set W with addition and scalar multiplication forms a vector space. However, if W is part of a larget set V that is already known to be a vector space, then certain axioms need not be verified for W because they are inherited from V.

Examples of Subspaces

1. A plane through the origin of R^3 forms a subspace of R^3. This is evident geometrically as follows: Let W be any plane through the origin and let u and v be any vectors in W other than the zero vector. Then u+v must lie in W because it is the diagonal of the parallelogram determined by u and v, and ku must lie in W for any scalar k because ku lies on a line through u. Thus, W is closed under addition and scalar multiplication, so it is a subspace of R^3.
2. A line through the origin of R^3 is also a subspace of R^3. It is evident geometrically that the sum of two vectors on this line also lies on the line and that a scalar multiple of a vector on the line is on the line as well. Thus, W is closed under addition and scalar multiplication, so it is a subspace of R^3

Modules

A vector space M over a field R is a set of objects called vectors, which can be added, subtracted and multiplied by scalars (members of the underlying field). Thus M is an abelian group under addition, and for each r ∈ R and x ∈ M we have an elementrx ∈ M. Scalar multiplication is distributive and associative, and the multiplicative identity of the field acts as an identity on vectors. Formally,

$$r(x+y) = rx+ry, (r+s)x = rx+sx, (rs)x = r(sx), 1x = x$$

For all $x, y \in M$ and, $s \in R$. A module is just a vector space over a ring. The formal definition is exactly as above, but we relax the requirement that R be a field, and instead allow an arbitrary ring. We have written the product rx with the scalar r on the left, and technically we get a left R-module over the ring R. The axioms of a right R-module are

$$(x+y)r = xr+yr, x(r+s) = xr+xs, x(rs) = (xr)s, x.1 = x$$

Examples of Modules

1. If M is a vector space over the field R, then M is an R-module.
2. Any ring R is a module over itself. Rather than check all the formal requirements, think intuitively: Elements of a ring can be added and subtracted, and we can certainly multiply r∈R by x∈R, and the usual rules of arithmetic apply.

Sub Modules

If N is a nonempty subset of the R-module M, we say that N is a Submodule of M (notation N≤M) if for every x,y ∈ N and r,s ∈R, we have rx+sy ∈ N. If M is an R-algebra, we say that N is a subalgebra if N is a submodule that is also a subring.

For example, if A is an abelian group (=Z-module), the submodules of A are the subsets closed under addition and multiplication by an integer (which amounts to addition also). Thus the submodules of A are simply the subgroups. If R is a ring, hence a module over itself, the submodules are those subsets closed under addition and also under multiplication by any r∈R, in other words, the left ideals. (If we take R to be a right R-module, then the submodules are the right ideals) We can produce many examples of subspaces of vector spaces by considering kernels and images of linear transformations. A similar idea applies to modules.

Comparison

The Properties of Modules that emphasize differences between Modules and Vector Spaces are as follows.

(1) In the case of vector spaces, every subspace has a complement. However, a sub-module of a module need not have a complement.

Example. The set of integers Z is a Z -module (a module over itself). The sub-modules of the Z-module Z are precisely the ideals of the ring Z and Z is a PID (Prove these properties!). Then the sub-modules of
Z are precisely the sets.

$$< n > = Z_n = \{zn : z \in Z\}$$

Thus, all nonzero submodules of Z are of the form Zu for some positive u ∈ Z So we see that any two nonzero sub-modules of Z have nonzero intersections, For if u, v > 0 then $0 \neq uv \in Zu \cap Zv$. Hence none of the sub-modules Zu, for $u \neq o\ or\ 1$, have complements.

(2) A vector space is finitely generated if and only if it has a finite basis. A sub-module of a finitely generated module need not be finitely generated.

Example Let the ring R =F[X] of all polynomials in infinitely many variables a finite sum, involves only finitely many variables. Then R is an R-module, and is finitely generated by the identity p(X) = 1.

Now consider the sub-module S of all polynomials with zero constant term. This module is generated by the variables themselves

$$S = < x_1, x_2, x_3, \dots\dots\dots >$$

However, S is not finitely generated by any finite set of polynomials. To see this we suppose that $\{p_1, \dots\dots p_n\}$ g is a finitely generating set for S. Then, for each k there exist polynomials $a_{k,1}(x), \dots\dots a_{k,n}(x)$ for which

$$x_k = \sum_{i=1}^{n} a_{k,i}(x) p_i(x). \quad (1)$$

Note that since $p_i(X) \in S$, it has zero constant term.

Since there are a finite number of variables involved in all of the $p_i(X)$'s. we can choose an index k for which $p_1(X)$; $p_n(x)$ do not involve x_k. For each $a_{k,j}(X)$ let us collect all terms involving x_k, and all terms not involving x_k.

$$a_{k,j} = x_k q_j(x) + r_j(x) \quad (2)$$

Where $q_j(x)$ is any polynomial in R and $r_j(x)$ does not involve x_k.
Now (1) and (2) yield

$$x_k = \sum_{i=1}^{n} \left(x_k q_j(x) + r_j(x) \right) p_i(x)$$
$$= x_k \sum_{i=1}^{n} q_j(x)\, p_i(x) + \sum_{i=1}^{n} r_j(x)\, p_i(x)$$

The last sum does not contain x_k, and so it must be 0. Hence, the rest sum must equal to 1, but this is not possible, since the $p_j(X)$'s have no constant terms. Hence S has no finite generating set.
(3) In vector spaces the set S={v}, consisting of a single nonzero vector v, is linearly independent. However, in a module, this need not be the case.
Example. The abelian group $z_n = \{1,2,3, \dots \dots . n-1\}$ is a Z -module, with scalar multiplication defined by za= (z.a) mod n, for all n ∈ Z and a ∈ Z_n. However, since na= 0 for all a ∈ Zn, we see that no singleton set {a} is linearly independent.
(4) In a vector space there are always a basis. But there are modules with no linearly independent elements, and hence with no basis. See the example in the previous item.
(5) In a vector space, a set S of vectors is linearly dependent if and only if some vectors in S is a linear combination of the other vectors in S. For arbitrary modules, this is not true.
Example. Consider the Z-module Z_2, consisting of all ordered pairs of integers. Then the ordered pairs (2,0) and (3,0) are linearly dependent, since 3(2,0) – 2(3,0) = (0,0) but neither one of these ordered pairs is linear combination (i.e. scalar multiple) of the other.
(6) In a vector space, a set of vectors is a basis if and only if it is a minimal spanning set, or equivalently, a maximal linearly independent set. For the modules the following is the best we can do in general.
Theorem. Let B be a basis for an R-module M. Then
(a) B is a minimal spanning set.
(b) B is a maximal linearly independent set.

That is in a module a minimal spanning set is not necessarily a basis and a maximal linearly independent set is not necessarily a basis. Such sets may not exist.
The Z-module in (3) is an example of a module that has no basis, since it has no linearly independent sets. But since the entire module is a spanning set, we claim that a minimal spanning set need not be a basis.
(7) There exist free modules with linearly independent sets that are not contained in a basis, and spanning sets do not contain a basis. This is to say that even free modules are not very much like vector spaces.
Example. The set Z×Z is a free module over itself with basis {(1, 1)}. To see this, observe that (1,1) is linearly independent, since (m, n)(1,1) = (0,0) implies (m, n) = (0,0) Also, (1,1) spans Z×Z, since (m, n) = (m, n)(1,1) But the sub-module Z×{0} is not free, since it has no linearly independent elements, and hence no basis. This follows from the fact that, if (n,0) ≠ (0,0), then, for instance (0,1)(n,0) = (0,0), and so {(n,0)} is not linearly independent.

Conclusion
However from the definition point of view, the concept of vector space and Modules looks like similar, but in Vector Space the role of scalar field and in Modules the role of scalar ring creates a big difference between the two concepts. Lots of properties as discussed in the paper differ due to this reason. Results of vector spaces are very much different from the results of modules as discussed in paper, So, the roles of Scalar in Vector Space and Modules play a very important role which creates drastic changes in the results.

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