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Asymmetric and symmetric properties of constant shape Bi-Weibull ROC curve described by Kullback-Leibler divergences

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Abstract
Receiver Operating Characteristic (ROC) Curve is used for assessing the ability of a biomarker/screening test to discriminate between nondiseased and diseased subject. In this paper, the parametric ROC curve is studied by assuming Constant Shape Bi-Weibull distribution to the biomarker values. The ROC model developed under this assumption is called Constant Shape Bi-Weibull ROC (WROC) model. Here, the research interest is to know how far the biomarker will make a distinction between diseased and non-diseased subjects when the gold standard is available using parametric WROC curve. The accuracy of diagnostic test is depends on two populations and their characteristics. The properties of WROC curve that explains the behavior of the WROC curve are also discussed. In order to explain WROC properties, the concept of Kullback Leibler Divergence (KLD) is used to study the symmetric and asymmetry properties of WROC Curve. To explain this phenomenon, KLD has been estimated using a simulation study and a real data set.

Keywords: Constant shape Bi-Weibull ROC curve, symmetry, asymmetry, Kullback-Leibler divergence, relative entropy

1. Introduction

The Receiver Operating Characteristic (ROC) Curve has made its landmark in classification problems which helps in assessing the performance of a test or diagnostic procedure. The ROC Curve is embedded by the two intrinsic measures Sensitivity (S_n) or True Positive Rate (TPR) and 1-Specificity (S_p) or False Positive Rate (FPR) along with its accuracy measure Area under the ROC Curve (AUC). The term ROC analysis was coined during II world war to analyze the radar signals [27]. The application of ROC Curve technique was promoted in diversified fields such as experimental psychology [12], industrial quality control [8] and military monitoring [29, 12] Was first to use the Gaussian model for estimating the ROC Curve [5]. Gave the maximum likelihood estimates for the ROC Curve parameters considering yes or no type responses and rating data. The importance of ROC Curve in medicine was due to [20] analyze the radiographic images [22]. Proposed a methodology to describe the scores of rating type by embedding the approach of [5, 14] Explained the importance and robustness of Binormal ROC Curve [21]. Proposed a unique type of ROC Curve where the decision variable is a strictly increasing function of the likelihood ratio, named it as the "Proper" ROC Curve. Earlier work on this Proper ROC Curve was due to [9] which take into the account of chi-square and gamma distributions [10]. Focused on identifying the optimal thresholds by proposing an exponential type ROC Curve. The ROC models based on Lomax and Gamma distributions was given by [7]. In later years, the estimation of parameters of ROC Curves was brought into the setup of regression frame work by [24]. Further different methodologies to estimate the ROC Curve by studying the effect of covariates where the data is of continuous type is given by [25]. Goodness of fit procedures and inferential aspects to test the accuracy measure and intrinsic measures of ROC Curve was given by [33, 11]. For more details on the methodological development of ROC Curves can be found in [32, 17].

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Recently we developed Functional Relationship between Brier Score and Area under the Constant Shape Bi-Weibull ROC Curve [19] and Confidence Intervals Estimation for ROC Curve, AUC and Brier Score under the Constant Shape Bi-Weibull Distribution [18]. The present work focuses on Properties of Constant Shape Bi-Weibull ROC curve using the parameter b. This paper is organized as follows: Section 2, provides the expected value of log likelihood ratio of proposed WROC Curve which is equivalent to the KLD; along with this the symmetric and asymmetric properties of proposed WROC Curve are also discussed. In Section 3, we discussed simulation study for proposed theory. Section 4, deals with validated of proposed theory based on real data. Finally conclusions are provided in Section 5.

2. A Constant Shape Bi-Weibull ROC Model and its AUC
In medical science, a diagnostic test result called a biomarker [15, 2] is an indicator for disease status of patients. The accuracy of a medical diagnostic test is typically evaluated by sensitivity and specificity. Receiver Operating Characteristic (ROC) curve is a graphical representation of the relationship between sensitivity and specificity. Hence the main issue in assessing the accuracy of a diagnostic test is to estimate the ROC curve.

Suppose that there are two groups of study subjects: diseased and nondiseased. Let S be a continuous biomarker. Assume that ROC analysis based on the True Positive Probability (TPP), $P(S|D)$, and False Positive Probability (FPP), $P(S|H)$, in fundamental detection problems with only two events and two responses [30, 12, 6].

According to Signal Detection Theory(SDT), we assume that there are two probability distributions of the random variables X and Y, one associated with the signal event s and other with the non signal event n [16]; these probability (or density) distributions of a given observation *x and y* are conditional upon the occurrence of *s* and *n* [6].

In the medical context, the signal event corresponds to the diseased group, and the nonsignal event to the nondiseased group [13]. If the cutoff value is *c*, corresponding to a particular likelihood ratio, the TPP and FPP are given by the following expressions [6]:

Let x, *y* be the test scores observed from two populations with (diseased individuals) and without (nondiseased individuals) condition respectively which follow Constant Shape Bi-Weibull distributions. The density functions of Constant Shape Bi-Weibull distributions are as follows,

$$f(x|D) = \frac{\beta}{\sigma_D} t^{\beta-1} e^{-\left[\frac{t^\beta}{\sigma_D}\right]}. \qquad (1)$$

and

$$g(y|H) = \frac{\beta}{\sigma_H} t^{\beta-1} e^{-\left[\frac{t^\beta}{\sigma_H}\right]}. \qquad (2)$$

The probabilistic definitions of the measures of ROC Curve are as follows:

$$\text{Sensitivity}(s_1) = P(S|D) = \int_t^\infty f(x|D)dx,$$

$$1 - \text{Specificity}(s_2) = P(S|H) = \int_t^\infty f(x|H)dx.$$

In this context, the (1-Specificity) and Sensitivity can be defined using equations (1) and (2) and are given in equations (3) and (4) respectively,

$$P(S|H) = y(t) = e^{-\left[\frac{t^\beta}{\sigma_H}\right]}. \qquad (3)$$

and

$$P(S|D) = x(t) = e^{-\left[\frac{t^\beta}{\sigma_D}\right]}. \qquad (4)$$

The ROC Curve is defined as a function of (1-Specificity) with scale parameters of distributions and is given as,

$$ROC(t) = x(t) = y(t)^b, \qquad (5)$$

where $t = -[\sigma_H log y(t)]^{\frac{1}{\beta}}$ is the threshold and $b = \frac{\sigma_H}{\sigma_D}$.

The accuracy of a diagnostic test can be explained using the Area under the Curve (AUC) of an ROC Curve. AUC describes the ability of the test to discriminate between diseased and nondiseased populations. A natural measure of the performance of the classifier producing the curve is AUC. This will range from 0.5 for a random classifier to 1 for a perfect classifier. The AUC is defined as,

$$AUC = \int_0^1 y(t)^b dx(t). \qquad (6)$$

The closed form of AUC is as follows

$$AUC = \frac{1}{1+b}. \qquad (7)$$

Now, we shall discuss some of the properties of W*ROC* curve.

2.1. Properties and Characteristics of WROC curve
Once the ROC curve is plotted, it is important to study the properties of it, in order to highlight some key understanding from the plot. It is known that a typical parametric ROC curve must satisfy the basic three properties viz. monotonicity, invariance to monotone increasing transformation and the slope defined at a particular threshold t [17]. Recently, [16] have given a quite interesting property known as asymmetry property of the ROC curve for a few ROC models viz. Bi-Exponential, Bi-Normal and Bi-Gamma. In this section, we have more generally discussed the properties satisfied by a parametric ROC curve.

Result: 1 (Proper ROC curve): The concavity property of *ROC* curve implies Proper. A ROC curve is said to be a proper *ROC* curve if it never crosses the chance line (the line connecting the co-ordinates $(0, 0)$ and $(1, 1)$). Otherwise, *TPR* is a strictly increasing function over the range of all possible *FPR*.

Proof: Consider any two points x and y (say) where 0< x, y <1 on the *FPR*.

By the definition of concavity, the line segment connecting the point on the *ROC* curve parallel to x and y never lies above the curve. If we take the extreme point that is the above Figure shows x=0 and y=1, it becomes the chance line which never lies above the curve. Hence we have proved that the concavity property of *ROC* curve implies it is also proper.

1. WROC curve is monotonically increasing in nature for $\sigma_D > \sigma_H$

Proof: Since, the first derivative of WROC curve with respect to *x(t)*is positive that is

$$\frac{dROC(t)}{dy(t)} = b\,y(t)^{b-1} > 0,$$

WROC curve is monotonically increasing in nature.

Aliter

Consider two FPR values $y(t_1)$and $y(t_2)$such that $y(t_1) < y(t_2)$.Now raising the power **b**, the inequality remains the same and hence

$$y(t_1)^b < y(t_2)^b.$$

Hence, the WROCcurve is monotonically increasing.

2. WROC curve is concave when $\sigma_D > \sigma_H$ and proper as long as it is concave

Proof: The second derivative of *ROC (t)* is given by

$$\frac{d^2ROC(t)}{d^2y(t)} = b(b-1)y(t)^{b-2} < 0,$$

The ratio $b = \frac{\sigma_H}{\sigma_D}$will always be less than one since we assumed that $\sigma_D > \sigma_H$and hence the term (b-1) <0 and $y(t)^{b-2} > 0$ since 0≤ y(t)≤1. On the whole, we will get

$$\frac{d^2ROC(t)}{d^2y(t)} = b(b-1)y(t)^{b-2} < 0,$$

Hence, WROC curve is concave in nature and by using Result 1, it is also proper.

Though normal distribution is thought to fit many real world datasets [15], it is not concave in nature in (0,1) that is the Bi-Normal ROC curve may lie below the chance line which in turn reduces the AUC. Therefore, we prefer a model that estimates ROC curve for biomarker which is concave in nature.

3. The slope of the WROC curve is $\frac{dx}{dy} = \frac{f(t|D)}{g(t|H)}$

Proof: The slope of an ROC Curve can be defined in three ways [1]: first, as the tangent at a particular point on the ROC Curve corresponding to a test value *x*, (That is tangent(*x*)); Second, the slope between origin 0 (That is point (0,0)) and the point on the ROC Curve corresponding to the a test value (That is slope(0-*x*)); and Third, the slope between two points on the ROC Curve corresponding to the test values *x* and *y* (That is slope(*x*-*y*)).

First, the tangent at a point *x* on the ROC Curve, That is, tangent(*x*) corresponds to the likelihood ratio (LR) for a

single test value corresponding to that point on the ROC Curve for a continuous test, That is, LR(*x*).Here, the LR is defined as the ratio between the probability of a defined test result given the presence of condition and the probability of the same result given the absence of condition. If a test generates results on a continuous scale, then a likelihood ratio can theoretically be defined for each test value *x* as

L(X) = {PDF of continuous test in variable x as *f(x/s)* in the presence of condition}/{ PDF of continuous test in variable x as *f(x/n)* in the presence of condition}

That is

$$L(x) = \frac{f(x|D)}{f(y|H)}. \tag{8}$$

Now, using equations (1) and (2), the slope of ROC Curve which is called as likelihood ratio can be defined as follows

$$\frac{f(x|D)}{f(y|H)} = \frac{dP(S|D)}{dP(S|H)} = b\exp\left(\left[\frac{t^\beta}{\sigma_H}\right] - \left[\frac{t^\beta}{\sigma_D}\right]\right). \tag{9}$$

The inverse slope of ROC Curve is defined as

$$\frac{f(x|H)}{f(x|D)} = \frac{dP(S|H)}{dP(S|D)} = \frac{1}{b}\exp\left(\left[\frac{t^\beta}{\sigma_D}\right] - \left[\frac{t^\beta}{\sigma_H}\right]\right). \tag{10}$$

4. WROC curve is TPP asymmetric

Proof: The Kullback - Leibler Divergence (KLD) is an essential equation of information theory that quantifies the proximity of two probability density functions and also based on the likelihood theory [4]. To study and interpret the WROC Curve, the information measure KLD has been used to identify the closeness between both distributions of diseased and non-diseased populations.

A KLD can be interpreted as a kind of distance between probability distributions [4], although the asymmetry in its arguments (apart from some special cases) clearly indicates it is not a distance in the Euclidian sense. We will work in natural logarithms, so the KLDs are denominated see [13]. For a discussion of measures of distance between distributions as used in summarizing ROC curves, see [26].

For a continuous indicator variables X and Y we denote pdfs *f(x)* (for diseased) and *g(y)* (for nondiseased) follow Constant Shape Bi-Weibull distribution, and the density functions are as follows,

$$f(x|D) = \frac{\beta}{\sigma_D} x^{\beta-1} e^{-\left[\frac{x^\beta}{\sigma_D}\right]}. \tag{11}$$

and

$$g(y|H) = \frac{\beta}{\sigma_H} y^{\beta-1} e^{-\left[\frac{x^\beta}{\sigma_H}\right]}. \tag{12}$$

Then the Constant Shape Bi-Weibull KLDs using equations (11) and (12) are $KL(f,g)$denote the *KL* divergence between the distribution of diseased and nondiseased population with $f(x)$as the comparison distribution and $g(y)$as the reference distribution [8] as

$$KL(f,g) = E_f\left(ln\left[\frac{f(x)}{g(y)}\right]\right) = \int_D f(x)ln\left[\frac{f(x)}{g(y)}\right]dz,$$

where z is the common range of x and y i.e. {LL=max[LL(y), LL(x₁)], UL=[min(UL(y), UL(x)] where LL-Lower Limit, UL – Upper limit. D is based on z, let us represent x and y in terms of z, as

$$KL(f,g) = \int_D f(z)ln\left[\frac{f(z)}{g(z)}\right]dz.$$

$$KL(f,g) = \int_0^\infty \frac{\beta}{\sigma_D}z^{\beta-1}e^{-\left[\frac{z^\beta}{\sigma_D}\right]} ln\left[\frac{\frac{\beta}{\sigma_D}z^{\beta-1}e^{-\left[\frac{z^\beta}{\sigma_D}\right]}}{\frac{\beta}{\sigma_H}z^{\beta-1}e^{-\left[\frac{z^\beta}{\sigma_H}\right]}}\right]dz.$$

$$= \left[\frac{\sigma_D}{\sigma_H}\right] - 1 - ln\left[\frac{\sigma_D}{\sigma_H}\right] = . \left\{\left[\frac{1}{b}\right] - 1 - ln\left[\frac{1}{b}\right]\right\} \qquad (13)$$

Similarly, $KL(g,f)$ denote the KL divergence between the distribution of non-diseased and diseased population with $g(y)$ as the comparison distribution and $f(x)$ as the reference distribution [8]. Then

$$KL(g,f) = E_g\left(ln\left[\frac{g(y)}{f(x)}\right]\right) = \int_D g(x)ln\left[\frac{g(y)}{f(x)}\right]dz,$$

where z is the common range of x and y i.e. {LL=max[LL(y), LL(x)], UL=[min(UL(y), UL(x)] where LL-Lower Limit, UL – Upper limit. D is based on z, let us represent x and y in terms of z, as

$$KL(g,f) = \int_D g(z)ln\left[\frac{g(z)}{f(z)}\right]dz.$$

$$KL(g,f) = \int_0^\infty \frac{\beta}{\sigma_H}z^{\beta-1}e^{-\left[\frac{z^\beta}{\sigma_H}\right]} ln\left[\frac{\frac{\beta}{\sigma_H}z^{\beta-1}e^{-\left[\frac{z^\beta}{\sigma_H}\right]}}{\frac{\beta}{\sigma_D}z^{\beta-1}e^{-\left[\frac{z^\beta}{\sigma_D}\right]}}\right]dz.$$

$$= \left[\frac{\sigma_H}{\sigma_D}\right] - 1 - ln\left[\frac{\sigma_H}{\sigma_D}\right] = [b - 1 - ln(b)]. \qquad (14)$$

Here $x, y > 0, \sigma_D, \sigma_H$ and $\beta > 0$. The indicator variable is calibrated so that the mean of the diseased distribution is larger than the mean of the nondiseased distribution, which requires $\sigma_D\Gamma\left(1+\frac{1}{\beta}\right) > \sigma_H\Gamma\left(1+\frac{1}{\beta}\right)$ which implies $\sigma_D > \sigma_H$. A graphical plot of TPR against FPR provides the ROC curve. Such curves are TPP-asymmetric [16].

It is found that $KL(f,g) > KL(g,f)$. Let $b = \left[\frac{\sigma_H}{\sigma_D}\right]$; since $b - 1 \geq ln(b)$ with equality only if b=1 [16] we have both $KL(f,g)$ and $KL(g,f) \geq 0$, with equality only if $f(x)$ and $g(y)$ are identical, as required. Now we observe that there are three possible conditions, namely

$KL(f,g) > KL(g,f)$ when b< 1,
$KL(f,g) = KL(g,f)$ when b= 1,
$KL(f,g) < KL(g,f)$ when b>1.

Hence, we have proved that, the WROC curve is TPP asymmetric. A numerical check for this condition is also presented in simulation studies.

5. WROC curve is invariance with respect to monotonically increasing transformation.

Remark 2.1: The entire computations of this paper were carried out using simulation study as well as real datasets. Initially, we provide a detailed interpretation about the ROC under the Constant Shape Bi-Weibull distribution.

3. Simulation study
Simulation study is conducted to demonstrate how the proposed methodology works in a simulated environment for explaining the behavior of ROC Curve and KLD's. **b,** which is the ratio of scale parameter(s) of Constant Shape Bi-Weibull distributions considered is in between 0 and 1. Further Table 1 depicts the ROC values by varying **b** values between zero and one.

Table 1: WROC values for Constant Shape Bi-Weibull Distribution

y(t)	b	AUC	ROC Values for Constant Shape Bi-Weibull Distribution									
			b=0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
0.1	0.1	0.9090	0.7943	0.6309	0.5011	0.3981	0.3162	0.2511	0.1995	0.1584	0.1258	0.1122
0.2	0.2	0.8333	0.8513	0.7247	0.6170	0.5253	0.4472	0.3807	0.3241	0.2759	0.2349	0.2167
0.3	0.3	0.7692	0.8865	0.7860	0.6968	0.6178	0.5477	0.4855	0.4305	0.3816	0.3383	0.318
0.4	0.4	0.7142	0.9124	0.8325	0.7596	0.6931	0.6324	0.5770	0.5265	0.4804	0.4383	0.4187
0.5	0.5	0.6666	0.9330	0.8705	0.8122	0.7578	0.7071	0.6597	0.6155	0.5743	0.5358	0.5176
0.6	0.6	0.6250	0.9502	0.9028	0.8579	0.8151	0.7745	0.7360	0.6993	0.6645	0.6314	0.6155
0.7	0.7	0.5882	0.9649	0.9311	0.8985	0.8670	0.8366	0.8073	0.7790	0.7517	0.7254	0.7125
0.8	0.8	0.5555	0.9779	0.9563	0.9352	0.9146	0.8944	0.8746	0.8553	0.8365	0.8180	0.8089
0.9	0.9	0.5263	0.9895	0.9791	0.9688	0.9587	0.9486	0.9387	0.9289	0.9191	0.9095	0.9047
1	0.95	0.5128	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

From Table 1 we observe that if b is lies between zero and one then the corresponding WROC and AUC values are obtained. Further Table 2 the KLD's values are found by varying b is in between zero and one.

Table 2: KLD values for Constant Shape Bi-Weibull Distribution

B	KL(f, g)	KL(g, f)	KL'(f, g)	KL'(g, f)
0.1	6.6974	1.4025	-90.000	-9.0000
0.2	2.3905	0.8094	-20.000	-4.0000
0.3	1.1293	0.5039	-7.7777	-2.3333
0.4	0.5837	0.3162	-3.7500	-1.5000
0.5	0.3068	0.1931	-2.0000	-1.0000
0.6	0.1558	0.1108	-1.1111	-0.6666
0.7	0.0718	0.0566	-0.6122	-0.4285
0.8	0.0268	0.0231	-0.3125	-0.2500
0.9	0.0057	0.0053	-0.1234	-0.1111
0.95	0.0013	0.0012	-0.0554	-0.0526

Hence from Table 2 it is observed that $KL(f, g) > KL(g, f)$. Also we obtained the derivatives $KL'(f, g)$ and $KL'(g, f)$. Further, the graphical plots are drawn for KLDs and its derivatives that are shown in Figure 2 and Figure 3 respectively. Also the graphical plot is drawn for Constant Shape Bi-Weibull ROC values that are shown in Figure 1.
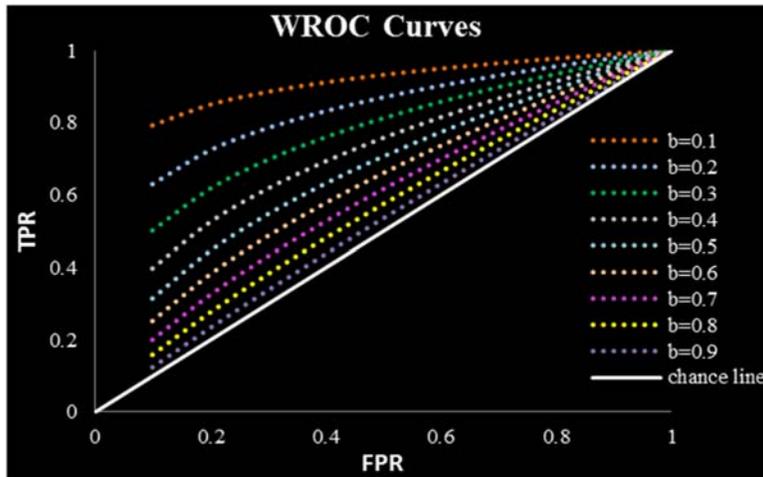

Fig 1: ROC curves for Constant Shape Bi-Weibull distribution

From Figure 1, the ROC Curves lies above the chance line. That is the curve approaches to the left hand corner and has a distance from the chance line that provides particular cutoff which establish the good accuracy of a diagnostic test.

Figure 2, shows the effect of $b = \left[\frac{\sigma_H}{\sigma_D}\right]$ on the behavior of the

KLDs as $KL(f, g) = \left\{\left[\frac{1}{b}\right] - 1 - \ln\left[\frac{1}{b}\right]\right\}$(Solid line),

and $KL(g, f) = [b - 1 - \ln(b)]$ (dashed line).

Another it is turns out to be easier to characterize $KL(f, g) > KL(g, f)$ portrayed in Figure 3 if we calculate the derivatives $KL'(f, g) = \left[\frac{b-1}{b^2}\right]$ and $KL'(g, f) = \left[\frac{b-1}{b}\right]$. Further, Figure 3 shows that for 0<b<1,

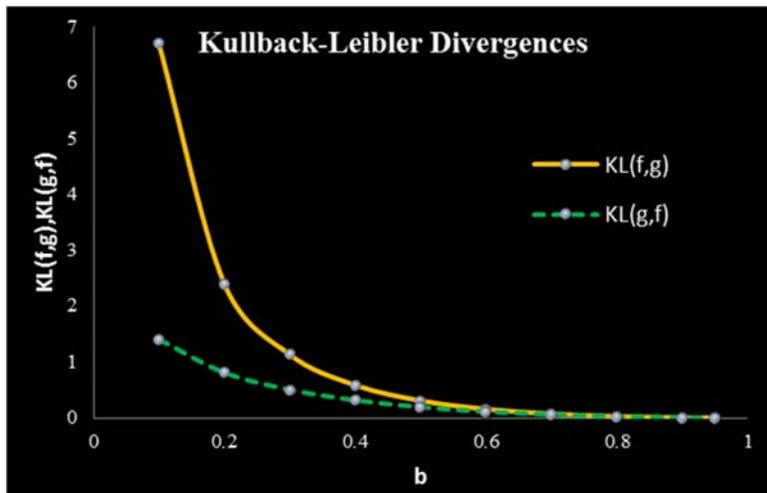$KL'(f, g) < KL'(g, f)$ describes the relationship between $KL(f, g)$ and $KL(g, f)$.

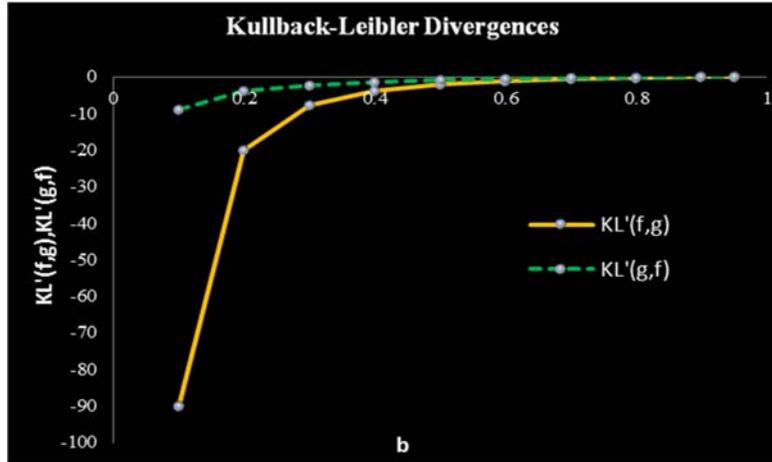**Fig 2:** KLDS $\{ KL(f, g), KL(g, f)\}$ for Constant Shape Bi-Weibull distribution

Fig 3: KLDS { $KL'(f, g), KL'(g, f)$ }for Constant Shape Bi-Weibull distribution

From the Figures 2 and 3 it can be seen that the corresponding ROC curve is TPP asymmetric. Also Bi-weibull ROC curves may be TPP asymmetric or TNP asymmetric. A Constant Shape Bi-Weibull ROC curve is always TPP asymmetric and $KL(f, g) > KL(g, f)$. For the Constant Shape Bi-Weibull ROC curve is always $KL'(f, g) < KL'(g, f)$.In this case the KLDs are equal only when $KL(f, g) = KL(g, f) = 0$ [10], indicating that (unlike the bi-Normal case) there is no symmetric curve that lies above the main diagonal of the Constant Shape Bi-Weibull ROC plot.

4. Illustration
The real data set namely German Breast Cancer Study [GBCS] extracted from [34]. The data describes primary node positive Breast Cancer recruited between July 1984, and December 1989 [34]. Data consists of 686 Observations, 16 variables. For this data we have to find KLD values and estimate parameters. This data consists of patients who are recurrences and who are censored. We have to know the patients with Cancer and patients without Cancer Out of these variables, Number of Progesterone Receptors (prog_recp), Number of Estrogen Receptors (estrg_recp) and Time to

Recurrence (rectime) are most influential variables for diagnose. In Table 3, the estimate of the parameter values of this data was reported along with the AUC and KLDs values.

Table 3: AUC and KLD values for German Breast Cancer Study Data

Test Variable(s)	b	AUC	KL(f,g)	KL(g,f)
Time to Recurrence	0.307190	0.765	1.075029	0.48748
Number of Progesterone Receptors	0.594896	0.627	0.161597	0.114264
Number of Estrogen Receptors	0.745201	0.573	0.047819	0.039302

From Table 3, it is observe that if b value is less than one, the corresponding KLDs values which show the relation $KL(f, g) > KL(g, f)$. So that, the WROC curve for German Breast Cancer Study Data is TPP asymmetric. Further Figure 4 depicts the ROC Curve for German Breast Cancer Study Data Using Constant Shape Bi-Weibull distribution.
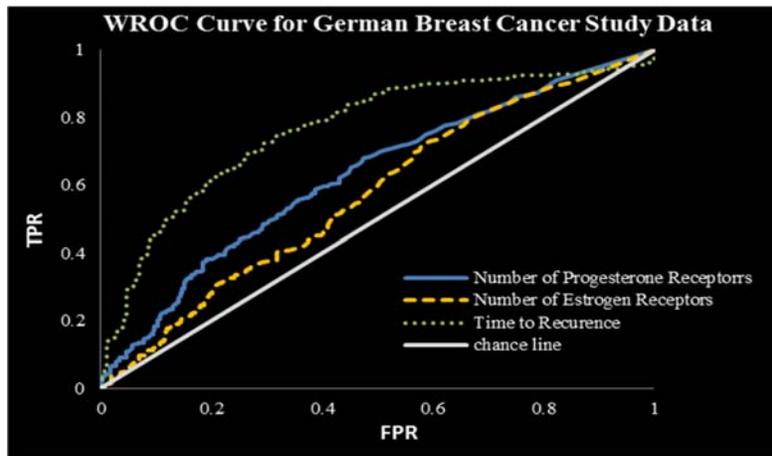

Fig 4: ROC Curve for German Breast Cancer Study Data distribution

From Figure 4, it is clear that ROC Curves lies above the chance line. That is the curve approaches to the left hand corner and has a distance from the chance line that provides

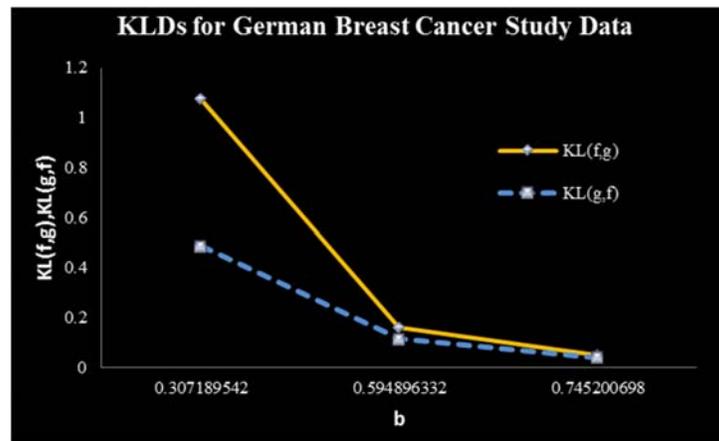particular cutoff which establish the good accuracy of a diagnostic test.

Fig 5: KLDS { $KL(f,g), KL(g,f)$} for German Breast Cancer Study Data

From the Figure 5 we can see that corresponding ROC curve is TPP-asymmetric. A Constant Shape Bi-Weibull ROC curve is always TPP-asymmetric and $KL(f,g) > KL(g,f)$ using German Breast Cancer Study Data. In this case the KLDs are equal only when $KL(f,g) = KL(g,f) = 0$, which indicating that (unlike the bi-Normal case) there is no symmetric curve that lies above the main diagonal of the Constant Shape Bi-Weibull ROC plot using German Breast Cancer Study Data.

5. Conclusion
The present work is carried out to characterize the symmetry properties of Constant Shape Bi-Weibull ROC curve in terms of the Kullback-Leibler divergences (KLDs) between the diseased and nondiseased distributions of risk scores. A Simulation study and illustrative example is also to explain the concepts. A Constant Shape Bi-Weibull ROC curve is always increasing, concave, and *TPP*-asymmetric and $KL(f,g) > KL(g,f)$ with KLDs derivatives $KL'(f,g) < KL'(g,f)$. In this case the KLDs are equal only when $KL(f,g) = KL(g,f) = 0$ [16], indicating that (unlike the bi-Normal case) there is no symmetric curve that lies above the main diagonal of the Constant Shape Bi-Weibull ROC plot. Our work so far, relating to continuous parametric ROC curves, indicates the following. First, although the KLD is usually not a symmetric quantity, it is noteworthy that for an ROC curve based on *f*(*x*) (for diseased) and *g*(y) (for nondiseased) that is symmetric about the negative diagonal, $KL(f,g) = KL(g,f)$ [3]. Second, although the lack of symmetry of the KLD has been referred to as a nuisance in applications [28], in this particular study we find that the asymmetry of the KLDs usefully characterizes the asymmetry of Constant Shape Bi-Weibull ROC curves.

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