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# A mathematical model on the two phase coronary blood flow through arterioles during angina

**Sunita Mishra, V Upadhyay, AK Agrawal and PN Pandey**

**Abstract**
In the present paper we have formulated the coronary blood flow in heart. Through arterioles the coronary circulation consists of the blood vessels that supply blood to remove blood from the heart muscle. The blood enters coronary arterioles from coronary artery. Upadhyay V. and Pandey P.N. have considered that the blood flow as two phase, one of red blood cells and other is plasma. They have also applied the Herschel Bulkley Non-Newtonian model in bio fluid mechanical set-up. We have collected a clinical data case of angina for hematocrit v/s blood pressure. The graphical presentation for particular parametric value is much closed to the clinical observation. The overall presentation is in tensorial form and solution technique adopted is analytical as well as numerical. The role of hematocrit is explicit the determination of blood pressure in the case of coronary heart disease with special reference to angina.

**Keywords:** Coronary blood flow, arterioles Herschel Bulkley, hematocrit, angina, non – Newtonian model, circulatory system

## Introduction
The human heart is a muscular organ containing four chambers that is situated just to the left of the midline of the thoracic cavity. The upper two chambers atria are divided by a wall like structure called the intertribal septum. The lower two chambers verticals are divided by a similar structure called the inter ventricular septum. Between each atrium and ventricle, valves allow blood to flow in one direction, preventing backflow. The wall of the heart has three layers –The Epicardium, myocardium, endocardium [1]. Blood low oxygen and high in carbon dioxide enters the right side of the heart and is pumped into the pulmonary circulation. After oxygenation into the lungs and some removal of carbon dioxide, it returns to the left side of the heart. The left ventricle pumps blood out of the heart to the rest of body.

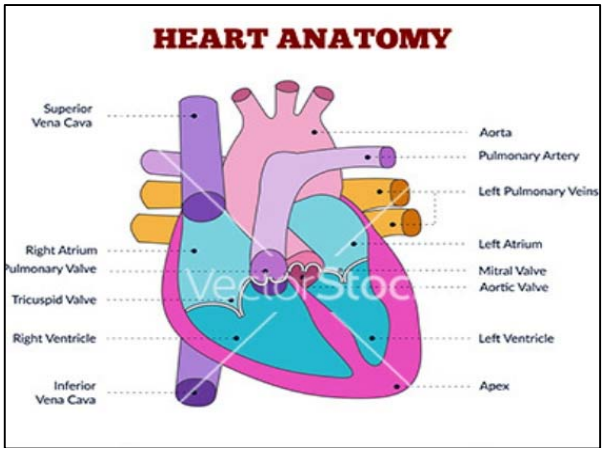



**Fig 1:** The Heart

Blood is a complex fluid consisting of particulate solids suspended in a non- Newtonian fluid. The particulate solids are red blood cells {RBCs}, white blood cells {WBCs} and platelets. The fluid is plasma, which itself is a complex mixture of proteins and other intergradient in an aqueous base 5O% of plasma and 45% of the blood cells and 45% of the blood is RBCs and there is a few parts of the other cells. Which are ignorable So one phase of the blood is plasma and second phase of the blood is RBCs. Two phase coronary blood flow is study of measuring the blood pressure if hemoglobin known. The percentage of volume covered by blood cells in the whole blood is called hematocrit.

The coronary circulation consists of the blood vessels that supply blood to and remove blood from the heart muscle itself. Although blood fills the chambers of the heart or myocardium is so think that is requires coronary blood vessels to deliver blood deep into the myocardium. The vessels that supply blood high in oxygen to the myocardium are known as coronary arteries [2]. These arteries when healthy are capable of auto regulation to maintain coronary blood flow at levels appropriate to the needs of the heart muscle.

Other blood vessels include arteries, veins, venules and capillaries. The structure of an arteriole is:- Arterioles are tiny branches of arteries that lead to capillaries. Arterioles are under the control of the sympathetic nervous system, and constrict and dilate to regulate blood flow. Arterioles, as a group, are the most highly regulated blood vessels in the body and contribute the most to overall blood pressure. Arterioles respond to a wide variety of chemical and electrical messages and are constantly changing size to speed up or slow down blood flow. Arterioles are an integral part of the circulatory system, which is a closed system in the sense that blood does not enter or leave the system during its journey from the heart, to the body, and back again. In such a system, a continuous flow of the same liquid can be pumped through the loop again and again.

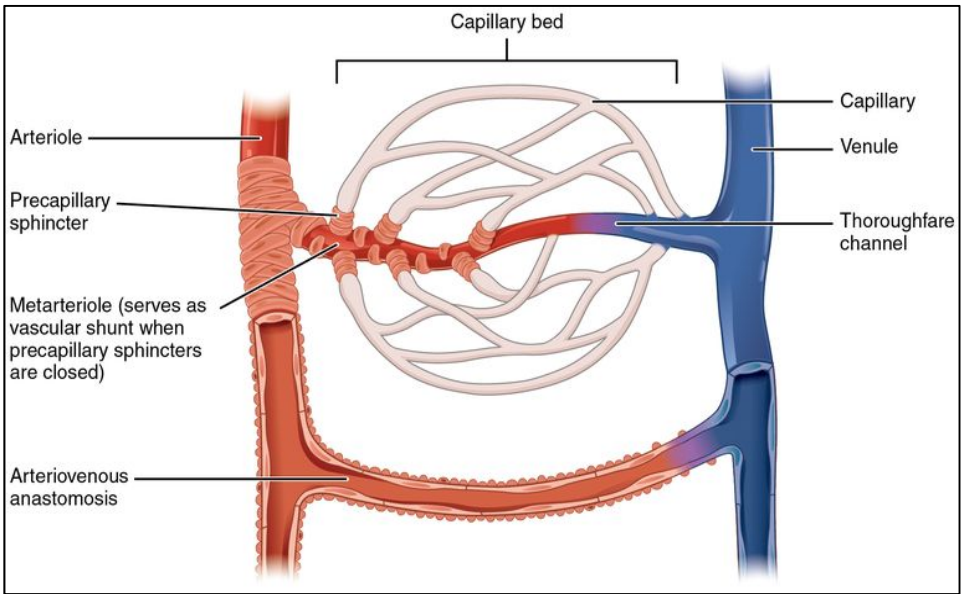



**Fig 2:** structure of Arterioles

Angina is chest pain or discomfort that occurs if an area of your heart muscle doesn't get oxygen rich blood. generally due to obstruction or spasm of coronary arteries [3]. The main cause of angina pectoris is coronary artery disease. Dueto atherosclerosis of the arteries feeding the heart. In some cases angina can be extremely serious and has been know to cause death. People that suffer from average to severe cases of angina have an increased percentage of death before the age of 55, usually around 6O%. There are three types of angina-

1. **Stable Angina:** Also known as effort angina this refers to the more common understanding of angina related to myocardial ischemia. Typical presentation of stable angina is that of chest discomfort ad associated symptoms precipitated by some activity [running, walking etc.] with minimal or non-existent symptoms at rest or with administration of sublingual nitroglycerin [4].
2. **Unstable Angina:** Unstable angina [UA] is defined as angina pectoris that changes or worsens [5]. It has at least of these three features-
   - It occurs at rest (or with minimal exertion ), usually lasting >10 min;
   - It is severe and new on set[i.e.; within the prior 4-6 weeks];
   - It occurs with a crescendo pattern.
3. **Micro vascular Angina:** Micro vascular angina or Angina syndrome X is characterized by angina like chest pain but it appears to be the result of spasm in the tiny lood vessels of the heart, arms and legs [6]

Chandra Harish, Upadhyay V., A. Agrawal A. K. Study in A Mathematical Model on the two phase Renal systolic blood flow in Arterioles with special reference to Diabetes. They applied the Herschel Bulkley. Non – Newtonian model. They collected a clinical data in case of diabetes for Hematocrit v/s Blood Pressure [9]. Singh J.P., Agrawal A. K, Upadhyay V. study in A Mathematical Model an Analysis of two phase Hepatic blood flow through Arteriors with special reference to Hepatitis A. They applied the Herschel Bulkley. Non – Newtonian model in Bio-fluid physiological are investigated. They collected a clinical data in case of Hepatitis A for

Hematocrit v/s Blood Pressure [10]. Srivastava Manoj, Upadhyay V., Agrawal A. K. Study in A Mathematical Model on the two phase pulmonary blood flow in Lungs with special reference to Asthma. They applied the Herschel Bulkley. Non – Newtonian model. They collected a clinical data in case of Asthma for Hematocrit v/s Blood Pressure [11]. A simple survey when Hematocrit increased, blood pressure is also increased, therefore the Hematocrit is proportional to the blood pressure. The graph between blood pressure and hematocrit is nearly closed to Herschel Bulkley model.

**Basic Bio-fluid equation for two phase blood flow**

Let us the problem of coronary blood flow is deferent form the problems in cylindrical tube and select generalized three dimensional orthogonal curvilinear coordinate system. Briefly descried as $E^3$ called as Euclidean space. The bio-physical laws thus expressed fully hold good in any co-coordinate system. Which is a computation for the truthfulness of the laws [13]. Blood is mixed fluid mainly there are two phases in blood. The first phase is plasma, while the other phase is that of red blood cells are enclosed with a semi permeable membrane whose density is grater than that of plasma. [14] Thus, blood can be considered as a homogeneous mixture of two phases

**Equation of continuity for two phase blood flow-**

The flow of blood is effected by the presence of blood is effected by the directly proportional to the volume occupied by blood cells. Let the volume portion covered by blood cells in unit volume be X, this X is replaced by H/100, where H is the Hematocrit the volume percentage of blood cells. Then the volume portion covered by the plasma will be 1-X, If the mass ratio of blood cells to plasma is r then clearly.

$$r = \ldots\frac{X\,\rho_c}{(1-X)\rho_p} \quad \ldots. (1)$$

Where $\rho_c$ and $\rho_p$ are densities of blood cells and blood plasma respectively. Usually this mass ratio is not a constant, even then this may be supposed to constant in present context [13]. The both phase of blood i.e. blood Cells and plasma move with the common velocity. Campbell and Pitcher have presented a model for this situation. According to this model, we consider the two phase of blood separately.

$$\frac{\partial(X\,\rho_p)}{\partial t} + (X\rho_c V^i),i = 0 \quad \ldots.(2)$$

$$\text{And } \frac{\partial(1-X)\rho_p}{\partial t} + (1\text{-}X)\rho_p V^i),i = 0 \quad \ldots.(3)$$

Where, V is the common velocity of two phase blood cells and plasma. If we define the uniform density of the blood $\rho_m$ as follow-

$$\frac{(1+r)}{\boldsymbol{\rho_m}} = \frac{1}{\rho_c} + \frac{1}{\rho_p} \quad \ldots (4)$$

Then equation (2) and (3) can be combined together as follow,

$$\frac{\partial\rho_m}{\partial t} + (\rho_m V^i),i = 0 \quad \ldots(5)$$

**Equation of motion for two phase blood flow**

The hydro dynamical pressure **p** between two the phase of the blood can be supposed to be uniform because the both phases i.e. blood cells and plasma are always in equilibrium state in blood(14), Taking viscosity coefficient of blood cells to be $\eta_c$ and applying the equation of motion for two phase of blood cells as follows-

$$X\rho_c \frac{\partial v^i}{\partial t} + (X\rho_c V^j)V^i{}_j = -X_{P,j}g^{ij} + X\eta_c(g^{jk}v^i{}_k)_{,\,ij} \quad \ldots.(6)$$

Taken the viscosity coefficient of plasma to be $\eta_p$, The equation of motion for plasma will be as follows-

$$(1\text{-}X)\rho_p \frac{\partial V^i}{\partial t} + \{(1\text{-}X)\,\rho_p\,V^i\}\,V_j{}^i = -(1\text{-}X)p\,{}_jg^{ij} + (1\text{-}X)\eta_c(g^{ij}\,V^i{}_{,k})_j \quad (7)$$

Now adding equation (6) and (7) and using relation (4), the equation of the motion for blood flow with the both phases will be as follow-

$$\rho_m \frac{\partial V^i}{\partial t} + (\rho_m V^i)\,V_j{}^i = -p,\,{}_jg^{ij} + \eta_m\,(g^{ij}\,V^i{}_{,k})_j \quad \ldots.(8)$$

where $\eta_c$ is viscosity of blood cells, $\eta_p$ viscosity of plasma and $\eta_{lm}=X\eta_c + (1\text{-}X)\eta_p$ is the viscosity coefficient of blood as a mixture of two phases. As the viscosity of blood flow decreases, the velocity of blood increases. The velocity of blood decreases successively because of the fact that arterioles. These vessels are relativity far enough from the heart. Hence the pumping of the heart on these vessels is relativity low. Secondly these vessels relativity narrow down more rapidly in this situation, the blood cells line up on the axis to build up rouleaux. Hence a yield stress is produced. Though this yield stress is very small, even then the viscosity of blood is increased nearly ten time. because of the fact that arterioles. These vessels are relativity far enough from the heart. Hence the pumping of the heart on these vessels is relativity low. Secondly these vessels relativity narrow down more rapidly in this situation, the blood cells line up on the axis to build up rouleaux. Hence a yield stress is produced. Though this yield stress is very small, even then the viscosity of blood is increased nearly ten ti

**Mathematical Modeling**

As the velocity of Blood flow decreases, the viscosity of blood increases. The velocity of blood decreases successively. The Herschel Bulkley law holds good on the two, phase blood flow though arterioles. Whose constitutive equation is as follows-

$T = \eta_m e^n + T_p (T' \geq T_p)$

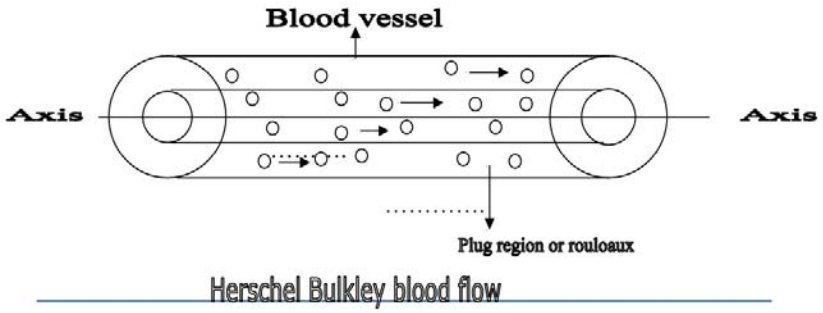
and $e = 0$ ( $T' < T_p$) where, $T_p$ is the yield stress.

When strain rate $e = 0 (T' < T_p)$

A core region is formed which flows just like a plug. Let the radius of the plug be $r_p$. The stress acting on the surface of plug will be $T_p$. Equating the forces acting on the plug, we get

$$P\pi r^2{}_p = T_p 2\pi r_p$$

$$r_p = 2\,\frac{T_p}{P} \quad \ldots.(9)$$




The constitutive equation for test part of the blood vessel is

$T = \eta_m e^n + T_p$
or $T - T_p = \eta_m e^n = T_e$

where, $T_e$ = effective stress
whose generalized form will be as follows
$T^{ij} = -pg^{ij} + T_e^{ij}$ where, $T_e^{ij} = \eta_m(e^{ij})^n$
While $e^{ij} = g^{ik} V_k^{\ i}$
Where the symbols have their usually meanings.
Now we describe the basic equations for therschel Bulkely blood flow as follows:-
Equation of Continuity-
$1/\sqrt{g}\sqrt{(gV^i)}_{,i} = 0$ …(10)
Equation of Motion-
$\rho_{m*}\partial V^i/\partial t + \eta_m V^j V^i_{\ j} = -T_{e,j}^{\ ij}$ ….(11)
Where all the symbols have their usual meaning.

**Solution and Discussion**
Since, the blood vessels are cylindrical. The above governing equations have to be transformed into cylindrical co-ordinates. As we know earlier:-

$x^1 = r, x^2 = \Theta, x^3 = z$

Matrix of metric tensor in cylindrical co-ordinates is as follows:-

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

While matrix of conjugate matrix tensor is as follow-

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Whereas the chritoffel's symbol of 2nd kind are as follow:-

$\{^1_{2\,2}\} = -r$, $\{^{\ 1}_{2\,2}\} = \{^1_{2\,2}\} = \frac{1}{r}$

Remaining others are zero.
The equation of continuity
$\frac{\partial v}{\partial z} = 0$

The equation of motion
r-componant $0 = -\frac{\partial p}{\partial z}, \Theta = component = 0$

z component $0 = -\frac{\partial p}{\partial z} + \frac{\eta_m}{r}[r(\frac{\partial Vz}{\partial r})^n]$
Here, this fact has been taken in view that the blood flow is axially Symmetric in arteries concerned, i.e.
$V_\Theta = 0$ and $V_r$, $V_z$ and p do not depend upon Θ.
We get $V_z = v(r)$ and $p = p(z)$ and

$0 = -\frac{dp}{dz} + \frac{\eta}{r}[r(\frac{dv}{dz})^n]$

Since, pressure gradient $-\frac{dp}{dz} = p$

$r(\frac{dv}{dz})^n = -pr^2/2\eta_m + A$, we apply

Boundary condition: at r = 0, v = $v_o$ then A = 0
⇨ $-\frac{dv}{dr} = (\frac{P^r}{2\eta_m})^{1/n}$
⇨ Replace r from r-$r_p$
$-\frac{dv}{dr} = \left(\frac{½pr - ½pr_p}{\eta_m}\right)^{1/n}$ ….(12)

Integratin g above666 equation (12) under the no slip boundary condition- v = 0 at r = R so as to get:
$V = \left(\frac{P}{2\eta_m}\right)^{1/n} \frac{n}{n+1}[(R-r_p)^{1/n+1} - (r-r_p)^{1/n+1}]$ …(13)

This is the formula for velocity of blood flow in arterioles.
Putting r = $r_p$ to get the velocity $V_p$ of plug flow as follows:-
$V_p = \frac{1}{1+n} = (P/2\eta_m)^{1/n}(R-r_p)^{1/n+1}$ ….(14)

Where the value of $r_p$ is taken from-(7)

**Result (Bio-physical interpretation)**
**Observation-** Hematocrit vs. blood pressure from an authorized Jabalpur hospital & Research centre By Dr. Umesh Agrawal Patient Name-Mr. Krishna kumar (Age-52 years old)

**Diagnosis- Coronary Artery Disease**

| Date | HB | B.P | 3 HB | Hematorit |
|---|---|---|---|---|
| 22/05/2016 | 17.6% | 130/80 mmhg | 3*17.6 | 52.8 |
| 25/05/2016 | 17.2% | 125/80 mmhg | 3*17.2 | 51.6 |
| 27/05/2016 | 16.5% | 115/80 mmhg | 3*16.5 | 49.5 |
| 30/05/2016 | 16.2% | 110/80 mmhg | 3*16.2 | 48.6 |
| 02/06/2016 | 15.2% | 120/90 mmhg | 3*15.2 | 45.6 |

The flow flux phase blood flow in coronary arteries.
$Q = \int_0^{rp} 2\pi r v_p dr + \int_{rp}^{R} 2\pi r v\, dr$

$$=\oint_0^{rp} 2\pi \frac{n}{n+1}\left(\frac{p}{2\eta}\right)^{1/n}[(R-r_p)^{1/n+1}]dr + \oint_0^{rp} 2\pi \frac{n}{n+1}\left(\frac{p}{2\eta}\right)^{1/n}[(R-r_p)^{1/n+1}-(R-r_p)^{1/n+1}]dr$$

using(12) and (14)

$$=\frac{2\pi n}{n+1}\left(\frac{p}{2\eta}\right)^{1/n}(R-r_p)^{1/n+1}[\frac{r^2}{2}]\,r_p+\frac{2\pi n}{(n+1)}\left(\frac{p}{2\eta}\right)^{1/n}[\frac{r^2}{2}(R-r_p)^{1/n+1}-\frac{r(r-r_p)^{\frac{1}{n}}}{\frac{1}{n}+2}+\frac{(r-r_p)^{\frac{1}{n}+3}}{(\frac{1}{n}+2)(\frac{1}{n}+3)}]r_p$$

$$=\frac{\pi n}{n+1}\left(\frac{p}{2\eta}\right)^{1/n}(R)^{\frac{1}{n}+3}[\frac{r^{\frac{2}{p}}}{R^2}(1-\frac{r^{\frac{2}{p}}}{R})^{1/n+1}+(1+\frac{r_p}{R})(1-\frac{r_p}{R})^{1/n+2}-\frac{2(1-\frac{r_p}{R})^{\frac{1}{n}+2}}{(\frac{1}{n}+2)}+\frac{2(1-\frac{r_p}{R})^{\frac{1}{n}+3}}{(\frac{1}{n}+2)(\frac{1}{n}+3)}]$$

Q = 425ml/min = 1, $r_{p=1/3}$

Gustafson, Daniel R. (1980)

$\eta_{p}$ = 0.0015 pascal-sec)

Glenn Elert (2010)

$\eta_{m}$ =0.035(pascal-sec)

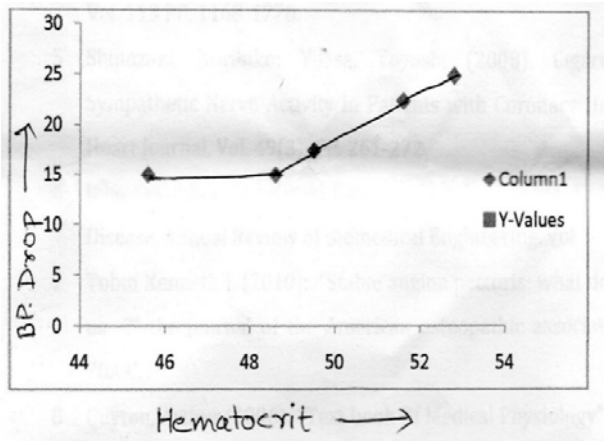
H = 49.62, P = 115

$\eta_{m} = \eta_{c} X + \eta_{p}(1-X)$ Where, X = H/ 100, X = 49.6/100= 0.496

$0.104587=(268.3113)^{1/n}\left(\frac{26n^3+33n^2+9}{6n^3+11n^2+6n+1}\right)$

we get n = 0.0112

| H | 52.8 | 51.6 | 49.5 | 48.6 | 45.6 |
|---|---|---|---|---|---|
| BP drop | 25 | 22.5 | 17.5 | 15 | 15 |




## Conclusion

A simple survey of the graph between blood pressure drop and hematocrit in cardiac patient shows that when hematocrit is increased the blood pressure drop is also increased.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