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## Certain fuzzy version of soft structures on soft g-modules

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**Abstract**

In this paper, we introduce two kinds of G-modules of a vector space with respect to soft structures, which are intersection fuzzy soft G-modules (IFSG-module) and union fuzzy soft G-module (UFSG-module). These new concepts show that how a soft set affects on a G-module of a vector space in the mean of intersection, union and inclusion of sets and thus, they can be regarded as a bridge among classical sets, fuzzy soft sets and vector spaces. We then investigate their related properties with respect to soft set operations, soft image, soft pre-image, soft anti image,  $\alpha$ -inclusion of fuzzy soft sets and linear transformations of the vector spaces. Furthermore, we obtain the relation between IFSG-modules and UFSG-modules and give the applications of these new G-modules on vector spaces.

**Keywords:** Soft set, IFSG-module, UFSG-module, fuzzy soft image, fuzzy soft anti image,  $\alpha$ -inclusion, trivial, whole

**1. Introduction**

Most of the problems in economics, engineering, medical science, environments etc. have various uncertainties. We cannot successfully use classical methods to solve these uncertainties because of various uncertainties typical for those problems. Hence some kinds of theories were given like theory of fuzzy sets [44], rough sets [17], i.e., which we can use as mathematical tools for dealing with uncertainties. However, all of these theories have their own difficulties which are pointed out in [15]. Soft set theory was introduced by Molodtsov [29] for modeling vagueness and uncertainty and it has been received much attention since Maji *et al* [27], Ali *et al* [6] and Sezgin and Atagun [34] introduced and studied operations of soft sets. Soft set theory has also potential applications especially in decision making as in [10,11,27]. This theory has started to progress in the mean of algebraic structures, since Aktas, and Cagman [5] defined and studied soft groups. Since then, soft substructures of rings, fields and modules [8], union soft substructures of near-rings and near-ring modules [35], normalistic soft groups [32] are defined and studied in detailed. Soft set has also been studied in the following papers [1, 2, 3, 25, 26, 45]. The theory of G-modules originated in the 20th century. Representation theory was developed on the basis of embedding a group G in to a linear group GL (V).

The theory of group representation (G module theory) was developed by Frobenius G [1962]. Soon after the concept of fuzzy sets was introduced by Zadeh [44] in 1965. Fuzzy subgroup and its important properties were defined and established by Rosenfeld [31] in 1971. After that in the year 2004 Shery Fernandez [36] introduced fuzzy parallels of the notions of G-modules. This study is of great importance since IFSG-modules and UFSG-modules show how a fuzzy soft set affect on a G-module of a vector space in the mean of intersection, union and inclusion of sets, so it functions as a bridge among classical sets, soft sets and vector spaces. In this paper, we introduce intersection fuzzy soft G-modules of a vector space that is abbreviated by IFSG- module and investigate its related properties with respect to fuzzy soft set operations. Then we give the application of fuzzy soft image, fuzzy soft pre image, upper  $\alpha$ -inclusion of fuzzy soft sets, linear transformations of vector spaces on vector spaces in the mean of IFSG-modules. Then we introduce union fuzzy soft G-module of a vector space that is abbreviated by UFSG-module and investigate its related properties and obtain a significant relation between IFSG-modules and UFSG-modules. Moreover, we apply soft pre image, soft anti-image, lower  $\alpha$ -inclusion of soft sets, linear transformations of vector spaces on these fuzzy soft G-modules. The work of this paper is organized as follows.

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In the second section as preliminaries, we give basic concepts of soft sets and fuzzy soft G-modules. In Section 3, we introduce IFSG-modules and study their characteristic properties. In Section 4, we give the applications of IFSG-modules and in the section 5, we introduce UFSG-modules and study some of their basic properties. Finally in section 6, applications of UFSG-modules are given.

## 2. Preliminaries

In this section as a beginning, the concepts of G-module<sup>[36]</sup> soft sets introduced by Molodsov<sup>[29]</sup> and the notions of fuzzy soft set introduced by Maji *et al.*<sup>[26]</sup> have been presented.

### 2.1 Definition<sup>[36]</sup>

Let  $G$  be a finite group. A vector space  $M$  over a field  $K$  (a subfield of  $C$ ) is called a  $G$ -module if for every  $g \in G$  and  $m \in M$ , there exists a product (called the right action of  $G$  on  $M$ )  $m.g \in M$  which satisfies the following axioms.

1.  $m.1_G = m$  for all  $m \in M$  ( $1_G$  being the identify of  $G$ )
2.  $m.(g.h) = (m.g).h$ ,  $m \in M$ ,  $g, h \in G$
3.  $(k_1 m_1 + k_2 m_2).G = k_1 (m_1.g) + k_2(m_2.g)$ ,  $k_1, k_2 \in K$ ,  $m_1, m_2 \in M$  &  $g \in G$ . In a similar manner the left action of  $G$  on  $M$  can be defined.

### 2.2. Definition<sup>[36]</sup>

Let  $M$  and  $M^*$  be  $G$ -modules. A mapping  $\emptyset: M \rightarrow M^*$  is a  $G$ -module homomorphism if

1.  $\emptyset(k_1 m_1 + k_2 m_2) = k_1 \emptyset(m_1) + k_2 \emptyset(m_2)$
2.  $\emptyset(gm) = g \emptyset(m)$ ,  $k_1, k_2 \in K$ ,  $m, m_1, m_2 \in M$  &  $g \in G$ .

### 2.3. Definition<sup>[36]</sup>

Let  $M$  be a  $G$ -module. A subspace  $N$  of  $M$  is a  $G$ -sub module if  $N$  is also a  $G$ -module under the action of  $G$ . Let  $U$  be a universe set,  $E$  be a set of parameters,  $P(U)$  be the power set of  $U$  and  $A \subseteq E$ .

### 2.4. Definition<sup>[29]</sup>

A pair  $(F, A)$  is called a soft set over  $U$ , where  $F$  is a mapping given by  $F: A \rightarrow P(U)$ .

In other words, a soft set over  $U$  is a parameterized family of subsets of the universe  $U$ .

Note that a soft set  $(F, A)$  can be denoted by  $F_A$ . In this case, when we define more than one soft set in some subsets  $A, B, C$  of parameters  $E$ , the soft sets will be denoted by  $F_A, F_B, F_C$ , respectively. On the other case, when we define more than one soft set in a subset  $A$  of the set of parameters  $E$ , the soft sets will be denoted by  $F_A, G_A, H_A$ , respectively. For more details, we refer to<sup>[11, 17, 18, 26, 29, 7]</sup>.

### 2.5. Definition<sup>[6]</sup>

The relative complement of the soft set  $F_A$  over  $U$  is denoted by  $F_A^r$ , where  $F_A^r: A \rightarrow P(U)$  is a mapping given as  $F_A^r(a) = U \setminus F_A(a)$ , for all  $a \in A$ .

### 2.6. Definition<sup>[6]</sup>

Let  $F_A$  and  $G_B$  be two soft sets over  $U$  such that  $A \cap B \neq \emptyset$ . The restricted intersection of  $F_A$  and  $G_B$  is denoted by  $F_A \Psi G_B$ , and is defined as  $F_A \Psi G_B = (H, C)$ , where  $C = A \cap B$  and for all  $c \in C$ ,  $H(c) = F(c) \cap G(c)$ .

### 2.7. Definition<sup>[6]</sup>

Let  $F_A$  and  $G_B$  be two soft sets over  $U$  such that  $A \cap B \neq \emptyset$ .

The restricted union of  $F_A$  and  $G_B$  is denoted by  $F_A \cup_R G_B$ , and is defined as  $F_A \cup_R G_B = (H, C)$ , where  $C = A \cap B$  and for all  $c \in C$ ,  $H(c) = F(c) \cup G(c)$ .

### 2.8. Definition<sup>[12]</sup>

Let  $F_A$  and  $G_B$  be soft sets over the common universe  $U$  and  $\psi$  be a function from  $A$  to  $B$ . Then we can define the soft set  $\psi(F_A)$  over  $U$ , where  $\psi(F_A): B \rightarrow P(U)$  is a set valued function defined by  $\psi(F_A)(b) = \cup \{F(a) \mid a \in A \text{ and } \psi(a) = b\}$ , if  $\psi^{-1}(b) \neq \emptyset$ ,  $= 0$  otherwise for all  $b \in B$ . Here,  $\psi(F_A)$  is called the soft image of  $F_A$  under  $\psi$ . Moreover we can define a soft set  $\psi^{-1}(G_B)$  over  $U$ , where  $\psi^{-1}(G_B): A \rightarrow P(U)$  is a set-valued function defined by  $\psi^{-1}(G_B)(a) = G(\psi(a))$  for all  $a \in A$ . Then,  $\psi^{-1}(G_B)$  is called the soft pre image (or inverse image) of  $G_B$  under  $\psi$ .

### 2.9. Definition<sup>[13]</sup>

Let  $F_A$  and  $G_B$  be soft sets over the common universe  $U$  and  $\psi$  be a function from  $A$  to  $B$ . Then we can define the soft set  $\psi^*(F_A)$  over  $U$ , where  $\psi^*(F_A): B \rightarrow P(U)$  is a set-valued function defined by  $\psi^*(F_A)(b) = \cap \{F(a) \mid a \in A \text{ and } \psi(a) = b\}$ , if  $\psi^{-1}(b) \neq \emptyset$ ,  $= 0$  otherwise for all  $b \in B$ . Here,  $\psi^*(F_A)$  is called the soft anti image of  $F_A$  under  $\psi$ .

**2.1.Theorem** <sup>[13]</sup>

Let  $F_H$  and  $T_K$  be soft sets over  $U$ ,  $F^r_H, T^r_K$  be their relative soft sets, respectively and  $\psi$  be a function from  $H$  to  $K$ . then,  
 i)  $\psi^{-1}(T^r_K) = (\psi^{-1}(T_K))^r$ ,  
 ii)  $\psi(F^r_H) = (\psi^*(F_H))^r$  and  $\psi^*(F^r_H) = (\psi(F_H))^r$

**2.10. Definition** <sup>[14]</sup>

Let  $F_A$  be a soft set over  $U$  and  $a$  be a subset of  $U$ . Then upper  $\alpha$ -inclusion of  $F_A$ , denoted by  $F_A^{\supseteq \alpha}$ , is defined as  $F_A^{\supseteq \alpha} = \{x \in A / F(x) \supseteq \alpha\}$ . Similarly,  
 $F_A^{\subseteq \alpha} = \{x \in A \mid F(x) \subseteq \alpha\}$  is called the lower  $\alpha$ -inclusion of  $F_A$ . A nonempty subset  $U$  of a vector space  $V$  is called a subspace of  $V$  if  $U$  is a vector space on  $F$ . From now on,  $V$  denotes a vector space over  $F$  and if  $U$  is a subspace of  $V$ , then it is denoted by  $U < V$ .

**3. IFSG-modules**

In this section, we first define intersection fuzzy soft G-modules of a vector space, abbreviated as IFSG-modules. We then investigate its related properties with respect to soft set operations.

Let  $G$  be a non-empty set. A fuzzy subset  $\mu$  on  $G$  is defined by  $\mu : G \rightarrow [0,1]$  for all  $x \in G$ .

**3.1. Definition**

Let  $G$  be a group. Let  $M$  be a  $G$ -module of  $V$  and  $A_M$  be a fuzzy soft set over  $V$ . Then  $A_M$  is called Intersection Fuzzy Soft G-module of  $V$  (IFSG-m), denoted by  $A_M \lesssim_l V$  if the following properties are satisfied  
 (IFSG-m<sub>1</sub>)  $A(ax + by) \geq A(x) \cap A(y)$   
 (IFSG-m<sub>2</sub>)  $A(\alpha x) \geq A(x)$ , for all  $x, y \in M, a, b, \alpha \in F$ .

**Example:** Let  $G = \{1, -1\}$ ,  $M = R^4$  over  $R$ . Then  $M$  is a  $G$ -module.

Define  $A$  on  $M$  by,

$$A(x) = \begin{cases} 1, & \text{if } x_i = 0 \forall i. \\ 0.5, & \text{if atleast } x_i \neq 0. \end{cases}$$

Where  $x = \{x_1, x_2, x_3, x_4\}; x_i \in R$ . Then  $A$  is a fuzzy soft  $G$ -Module.

**3.1. Proposition**

If  $A_M \lesssim_l V$ , then  $A(0v) \geq A(x)$  for all  $x \in M$ .

**Proof:** Since  $A_M$  is an IFSG-module of  $V$ , then  $A(ax+by) \geq A(x) \cap A(y)$  for all  $x,y \in M$  and since  $(M,+)$  is a group, if we take  $ay=-ax$  then, for all  $x \in M$ ,  
 $A(ax-ax) = A(0v) \geq A(x) \cap A(x) = A(x)$ .

**3.2. Proposition**

If  $A_{M_1} \lesssim_l V$  and  $B_{M_2} \lesssim_l V$ , then  $A_{M_1} \cap B_{M_2} \lesssim_l V$ .

**Proof:** Since  $M_1$  and  $M_2$  are  $G$ -modules of  $V$ , then  $M_1 \cap M_2$  is a  $G$ -module of  $V$ . By definition 2.6, let  $A_{M_1} \cap B_{M_2} = (A, M_1) \cap (B, M_2) = (T, M_1 \cap M_2)$ , where  
 $T(x) = A(x) \cap B(x)$  for all  $x \in M_1 \cap M_2 \neq \emptyset$ . Then for all  $x,y \in M_1 \cap M_2$  and  $\alpha \in F$ .  
 (IFSG-m<sub>1</sub>)  $T(ax+by) = A(ax+by) \cap B(ax+by) \geq (A(x) \cap A(y)) \cap (B(x) \cap B(y))$   
 $= (A(x) \cap B(x)) \cap (A(y) \cap B(y)) = T(x) \cap T(y)$ ,  
 (IFSG-m<sub>2</sub>)  $T(\alpha x) = A(\alpha x) \cap B(\alpha x) \geq A(x) \cap B(x) = T(x)$ .  
 There fore  $A_{M_1} \cap B_{M_2} = T_{M_1 \cap M_2} \lesssim_l V$ .

**3.2. Definition**

Let  $(A, M_1)$  and  $(B, M_2)$  be two IFSG-modules of  $V_1$  and  $V_2$  respectively, the product of IFSG-modules  $(A, M_1)$  and  $(B, M_2)$  is defined as  $(A, M_1) \times (B, M_2) = (Q, M_1 \times M_2)$ , where  $Q(x,y) = A(x) \times B(y)$  for all  $(x,y) \in M_1 \times M_2$ .

**3.1. Theorem**

If  $A_{M_1} \lesssim_l V$  and  $B_{M_2} \lesssim_l V$ , then  $A_{M_1} \times B_{M_2} \lesssim_l V_1 \times V_2$ .

**Proof:** Since  $M_1$  and  $M_2$  are  $G$ -modules of  $V_1$  and  $V_2$  respectively, then  $M_1 \times M_2$  is a  $G$ -module of  $V_1 \times V_2$ . By definition 3.2, let

$$\begin{aligned} A_{M_1} \times B_{M_2} &= (A, M_1) \times (B, M_2) \\ &= (Q, M_1 \times M_2), \text{ where } Q(x,y) = A(x) \times B(y) \text{ for all } (x,y) \in M_1 \times M_2. \end{aligned}$$

Then for all  $(x_1, y_1), (x_2, y_2) \in M_1 \times M_2$  and  $(\alpha_1, \alpha_2) \in F \times F$ ,  
 (IFSG-m<sub>1</sub>)  $Q\{(ax_1, by_1) + (ax_2, by_2)\} = Q(ax_1 + ax_2, by_1 + by_2)$   
 $= A(ax_1 + ax_2) \times B(by_1 + by_2)$   
 $\geq (A(x_1) \cap A(x_2)) \times (B(y_1) \cap B(y_2))$   
 $= Q(x_1, y_1) \cap Q(x_2, y_2)$

(IFSG- $m_2$ )  $Q((\alpha_1, \alpha_2)(x_1, y_1)) = Q(\alpha_1 x_1 + \alpha_2 y_1)$   
 $= A(\alpha_1 x_1) \times B(\alpha_2 y_2) \supseteq A(x_1) \cap B(y_2) = Q(x_1, y_1)$ .  
Hence  $A_{M_1} \times B_{M_2} = Q_{M_1 \times M_2} \prec_l V_1 \times V_2$ .

**3.3. Definition:** Let  $A_{M_1}$  and  $B_{M_2}$  be two IFSG-module's of  $V$ . If  $M_1 \cap M_2 = \{0_V\}$ , then the sum of IFSG-module's  $A_{M_1}$  and  $B_{M_2}$  is defined as  $A_{M_1} + B_{M_2} = T_{M_1 + M_2}$  where  $T(ax+by) = A(x)+B(y)$  for all  $ax+by \in M_1 + M_2$ .

**3.2. Theorem:** If  $A_{M_1} \prec_l V$  and  $B_{M_2} \prec_l V$  where  $M_1 \cap M_2 = \{0_V\}$ , then  $A_{M_1} + B_{M_2} \prec_l V$ .

**Proof:** Since  $M_1$  &  $M_2$  are  $G$ -modules of  $V$ , then  $M_1 + M_2$  is a  $G$ -modules of  $V$ . By definition:3.3, let  $A_{M_1} + B_{M_2} = (A, M_1) + (B, M_2) = (T, M_1 + M_2)$ , where  $T(ax+by) = A(x)+B(y)$  for all  $ax+by \in M_1 + M_2$ . It is obvious that since  $M_1 \cap M_2 = \{0_V\}$ , then the sum is well defined. Then for all  $ax_1 + by_1, ax_2 + by_2 \in M_1 + M_2$  and  $\alpha \in F$ ,  
 $T((ax_1 + by_1)+(ax_2 + by_2)) = T(ax_1 + ax_2 + by_1 + by_2)$   
 $= A(ax_1 + x_2) + B(by_1 + y_2)$   
 $\supseteq (A(x_1) \cap A(x_2)) + (B(y_1) \cap B(y_2))$   
 $= (A(x_1) + B(y_1)) \cap (A(x_2) + B(y_2))$   
 $= T(ax_1 + by_1) \cap T(ax_2 + by_2)$   
 $T(\alpha(x_1 + y_1)) = T(\alpha x_1 + \alpha y_1) = A(\alpha x_1) + B(\alpha y_1) \supseteq A(x_1) + B(y_1) = T(x_1 + y_1)$   
Thus,  $A_{M_1} + B_{M_2} \prec_l V$ .

**3.4. Definition**

Let  $A_M$  be an IFSG-module of  $V$ . Then,

- (i)  $A_M$  is said to be trivial if  $A(x) = \{0_V\}$  for all  $x \in M$ .
- (ii)  $A_M$  is said to be whole if  $A(x) = V$  for all  $x \in M$ .

**3.3. Proposition**

Let  $A_{M_1}$  and  $B_{M_2}$  be two IFSG-modules of  $V$ , then

- (i) If  $A_{M_1}$  and  $B_{M_2}$  are trivial IFSG-modules of  $V$ , then  $A_{M_1} \cap B_{M_2}$  is a trivial IFSG -module of  $V$ .
- (ii) If  $A_{M_1}$  and  $B_{M_2}$  are whole IFSG-modules of  $V$ , then  $A_{M_1} \cap B_{M_2}$  is a whole IFSG -module of  $V$ .
- (iii) If  $A_{M_1}$  is a trivial IFSG-module of  $V$  and  $B_{M_2}$  is a whole IFSG-modules of  $V$ , then  $A_{M_1} \cap B_{M_2}$  is a trivial IFSG -module of  $V$ .
- (iv) If  $A_{M_1}$  and  $B_{M_2}$  are trivial IFSG-modules of  $V$  where  $M_1 \cap M_2 = \{0_V\}$ , then  $A_{M_1} + B_{M_2}$  is a trivial IFSG-module of  $V$ .
- (v) If  $A_{M_1}$  and  $B_{M_2}$  are whole IFSG-modules of  $V$  where  $M_1 \cap M_2 = \{0_V\}$ , then  $A_{M_1} + B_{M_2}$  is a whole IFSG-module of  $V$ .
- (vi) If  $A_{M_1}$  is a trivial IFSG-module of  $V$  and  $B_{M_2}$  is a whole IFSG-modules of  $V$  where  $M_1 \cap M_2 = \{0_V\}$ , then  $A_{M_1} + B_{M_2}$  is a whole IFSG-module of  $V$ .

**Proof:** The proof is easily seen by definition2.6, definition3.3, definition3.4, theorem3.1 and theorem3.3.

**3.4. Proposition:** Let  $A_{M_1}$  and  $B_{M_2}$  be two IFSG-modules of  $V_1$  and  $V_2$  respectively. Then

- (i) If  $A_{M_1}$  and  $B_{M_2}$  are trivial IFSG-modules of  $V_1$  and  $V_2$  respectively, then  $A_{M_1} \times B_{M_2}$  is a trivial IFSG -module of  $V_1 \times V_2$ .
- (ii) If  $A_{M_1}$  and  $B_{M_2}$  are whole IFSG-modules of  $V_1$  and  $V_2$  respectively, then  $A_{M_1} \times B_{M_2}$  is a whole IFSG -module of  $V_1 \times V_2$ .

**Proof:** The proof is easily seen by definition3.2, definition3.4 and theorem3.3

**4. Applications of IFSG-modules:** In this section, we give the applications of soft image, soft pre image, upper  $\alpha$ -inclusion of fuzzy soft sets and linear transformation of vector spaces on vector space with respect to IFSG-modules.

**4.1. Theorem:** If  $A_M \prec_l V$ , then  $M_G = \{x \in M / A(x) = A(0_V)\}$  is a  $G$ -module of  $M$ .

**Proof:** It is obvious that  $0_V \in M_G$  and  $\emptyset \neq M_G \subseteq M$ . We need to show that  $ax+by \in M_G$  and  $\alpha x \in M_G$  for all  $x, y \in M_G$  and  $\alpha \in F$ , which means that  $A(ax+by) = A(0_V)$  and  $A(\alpha x) = A(0_V)$  have to be satisfied. Since  $x, y \in M_G$  and  $A_M$  is an IFSG-Module of  $V$ , then  $A(x) = A(y) = A(0_V)$ ,  $A(ax+by) \supseteq A(x) \cap A(y) = A(0_V)$ ,  $A(\alpha x) \supseteq A(x) = A(0_V)$  for all  $x, y \in M_G$  and  $\alpha \in F$ . Moreover, by Proposition3.1,  $A(0_V) \supseteq A(ax+by)$  and  $A(0_V) \supseteq A(\alpha x)$  which completes the proof.

**4.2. Theorem:** Let  $A_M$  be a fuzzy soft set over  $V$  and  $\alpha$  be a subset of  $V$  such that  $A(0_V) \supseteq \alpha$ . If  $A_M$  is an IFSG-module of  $V$ , then  $A_M^{\supseteq \alpha}$  is a  $G$ -module of  $V$ .

**Proof:** Since  $A(0_V) \supseteq \alpha$ , then  $0_V \in A_M^{\supseteq \alpha}$  and  $\emptyset \neq A_M^{\supseteq \alpha} \subseteq V$ . Let  $x, y \in A_M^{\supseteq \alpha}$ , then  $A(x) \supseteq \alpha$  and  $A(y) \supseteq \alpha$ . We need to show that  $x + y \in A_M^{\supseteq \alpha}$  and  $\alpha \in A_M^{\supseteq \alpha}$  for all  $x, y \in A_M^{\supseteq \alpha}$  and  $n \in F$ . Since  $A_M$  is an IFSG-module of  $V$ , it follows that  $A(ax + by) \supseteq A(x) \cap A(y) \supseteq \alpha \cap \alpha = \alpha$ .  
Furthermore,  $A(nx) \supseteq A(x) \supseteq \alpha$ , which completes the proof.

**4.3. Theorem:** Let  $A_M$  and  $T_W$  be fuzzy soft sets over  $V$ , where  $M$  and  $W$  are  $G$ -modules of  $\gamma$  and  $\Psi$  be a linear isomorphism from  $M$  to  $W$ . If  $A_M$  is an IFSG-Module of  $V$ , then so is  $\Psi(A_M)$ .

**Proof:** Let  $w_1, w_2 \in W$ . Since  $\Psi$  is a surjective linear transformation. Then there exists  $m_1, m_2 \in M$  such that  $\Psi(m_1) = w_1, \Psi(m_2) = w_2$ . Then  
 $(\Psi(A_M))(aw_1 + bw_2) = \cup\{A(m) : m \in M, \Psi(m) = aw_1 + bw_2\}$   
 $= \cup\{A(m) : m \in M, m = \Psi^{-1}(aw_1 + bw_2)\}$   
 $= \cup\{A(m) : m \in M, m = \Psi^{-1}(\Psi(aw_1 + bw_2)) = am_1 + bm_2\}$   
 $= \cup\{A(am_1 + bm_2) : m_i \in M, \Psi(m_i) = w_i, i = 1, 2\}$   
 $\supseteq \cup\{A(m_1) \cap A(m_2) : m_i \in M, \Psi(m_i) = w_i, i = 1, 2\}$   
 $= (\cup A(m_1) : m_1 \in M, \Psi(m_1) = w_1) \cap (\cup A(m_2) : m_2 \in M, \Psi(m_2) = w_2)$   
 $= (\Psi(A_M))(w_1) \cap (\Psi(A_M))(w_2)$

Now let  $\alpha \in F$  and  $w \in W$ . Since  $\Psi$  is a surjective linear transformation, there exists  $\tilde{m} \in M$  such that  $\Psi(\tilde{m}) = w$ . Then  
 $(\Psi(A_M))(\alpha w) = \cup\{A(m) : m \in M, \Psi(m) = \alpha w\}$   
 $= \cup\{A(m) : m \in M, m = \Psi^{-1}(\alpha w)\}$   
 $= \cup\{A(m) : m \in M, m = \Psi^{-1}(\Psi(\alpha \tilde{m})) = \alpha \tilde{m}\}$   
 $= \cup\{A(\alpha \tilde{m}) : \alpha \tilde{m} \in M, \Psi(\tilde{m}) = w\} = (\Psi(A_M))(w)$   
Hence,  $\Psi(A_M)$  is an IFSG-module of  $V$ .

**4.4. Theorem:**

Let  $A_M$  and  $T_W$  be fuzzy soft sets over  $V$ , where  $M$  and  $W$  are  $G$ -modules of  $\gamma$  and  $\Psi$  be a linear isomorphism from  $M$  to  $W$ . If  $T_W$  is an IFSG-Module of  $V$ , then so is  $\Psi^{-1}(T_W)$ .

**Proof:** Let  $m_1, m_2 \in M$ . Then  
 $\Psi^{-1}(T_W)(am_1 + bm_2) = T(\Psi(am_1 + bm_2))$   
 $= T(\Psi(am_1) + \Psi(bm_2))$   
 $\supseteq T(\Psi(m_1)) \cap T(\Psi(m_2))$   
 $= (\Psi^{-1}(T_W))(m_1) \cap (\Psi^{-1}(T_W))(m_2)$

Now let  $\alpha \in F$  and  $m \in M$ . Then,  
 $\Psi^{-1}(T_W)(\alpha m) = T(\Psi(\alpha m)) = T(\alpha \Psi(m)) \supseteq T(\Psi(m)) = \Psi^{-1}(T_W)(m)$   
Hence  $\Psi^{-1}(T_W)$  is an IFSG-module of  $V$ .

**4.5. Theorem:** Let  $V_1$  and  $V_2$  be two vector spaces and  $(A_1, M_1) \prec_l V_1, (A_2, M_2) \prec_l V_2$ .

If  $f: M_1 \rightarrow M_2$  is a linear transformation of vector spaces, then

- (i)  $f$  is surjective, then  $(A_1, f^{-1}(M_2)) \prec_l V_1$ ,
- (ii)  $(A_2, f(M_1)) \prec_l V_2$ ,
- (iii)  $(A_1, \ker f) \prec_l V_1$ .

**Proof:** (i) Since  $M_1 < V_1, M_2 < V_2$  and  $f: M_1 \rightarrow M_2$  is a surjective linear transformation, then it is clear that  $f^{-1}(M_2) < V_1$ . Since  $(A_1, M_1) \prec_l V_1$  and  $f^{-1}(M_2) < M_1, A_1(ax+by) \supseteq A(x) \cap A(y)$  and  $A_1(\alpha x) \supseteq A(x)$  for all  $x, y \in f^{-1}(M_2)$  and  $\alpha \in F$ . Hence  $(A_1, f^{-1}(M_2)) \prec_l V_1$ .

(ii) Since  $M_1 < V_1, M_2 < V_2$  and  $f: M_1 \rightarrow M_2$  is a vector space linear transformation, then  $f(M_1) < V_2$ . Since  $f(M_1) \subseteq M_2$ , the result is obvious by definition 3.1.

(iii) Since  $\ker f < V_1$  and  $\ker f \subseteq M_1$ , the rest of the proof is clear by definition 3.1.

**4.1. Corollary:** Let  $(A_1, M_1) \prec_l V_1, (A_2, M_2) \prec_l V_2$ . If  $f: M_1 \rightarrow M_2$  is a linear transformation, then  $(A_2, \{oM_2\}) \prec_l V_2$ .

**Proof:** By theorem: 4.5, (iii)  $(A_1, \ker f) \prec_l V_1$ , then  $(A_2, f(\ker f)) = (A_2, \{oM_2\}) \prec_l V_2$ ,  
By theorem 4.5 (ii).

**5. Union Fuzzy Soft G-modules**

In this section, we first define Union fuzzy soft  $G$ -modules of a Vector space, abbreviated by UFSG-modules. We then investigate its related properties with respect to soft set operations.

**5.1. Definition**

Let  $G$  be a group. Let  $M$  be a  $G$ -module of  $V$  and  $B_M$  be the fuzzy soft set over  $V$ . Then  $B_M$  is called Union fuzzy soft  $G$ -modules of  $V$ , denoted by  $B_M \lesssim_u V$  if the following properties are satisfied:  $B(ax+by) \subseteq B(x) \cup B(y)$  and  $B(\alpha x) \subseteq B(x)$ , for all  $x,y \in M$ , and  $\alpha \in F$ .

**5.1. Proposition**

If  $B_M \lesssim_u V$ , then  $B(0_V) \subseteq B(x)$  for all  $x \in M$ .

**Proof:** Since  $B_M$  is an UFSG-module of  $V$ , then  $B(ax+by) \subseteq B(x) \cup B(y)$  for all  $x,y \in M$  and since  $(M,+)$  is a group, if we take  $by=-ax$  then, for all  $x \in M$ ,

$$B(ax-ax) = B(0_V) \subseteq B(x) \cup B(x) = B(x), \text{ for all } x \in M.$$

In section 3, we showed that restricted intersection, the sum and product of two

IFSG-modules of  $V$  is an IFSG-modules of  $V$ . Now, we show that restricted union of two UFSG-modules of  $V$  with the following theorem:

**5.1. Theorem:** If  $A_{M_1} \lesssim_u V$  and  $B_{M_2} \lesssim_u V$ , then  $A_{M_1} \cup_{\mathcal{R}} B_{M_2} \lesssim_u V$ .

**Proof:** By definition 2.7, let  $A_{M_1} \cup_{\mathcal{R}} B_{M_2} = (A, M_1) \cup_{\mathcal{R}} (B, M_2) = (Q, M_1 \cap M_2)$ , where  $Q(x) = A(x) \cup B(x)$  for all  $x \in M_1 \cap M_2 \neq \emptyset$ . Since  $M_1$  &  $M_2$  are  $G$ -modules of  $V$ , then  $M_1 \cap M_2$  is a  $G$ -modules of  $V$ .

Let  $x,y \in M_1 \cap M_2$  and  $\alpha \in F$ . Then

$$Q(ax+by) = A(ax+by) \cup B(ax+by) \subseteq (A(x) \cup A(y)) \cup (B(x) \cup B(y))$$

$$= (A(x) \cup B(x)) \cup (A(y) \cup B(y)) = Q(x) \cup Q(y)$$

$$Q(\alpha x) = A(\alpha x) \cup B(\alpha x) \subseteq A(x) \cup B(x) = Q(x)$$

Therefore  $A_{M_1} \cup_{\mathcal{R}} B_{M_2} = Q_{M_1 \cap M_2} \lesssim_u V$ .

**5.2. Definition:** Let  $B_M$  be a UFSG-module of  $V$ . Then

(a)  $B_M$  is said to be trivial if  $B(x) = \{0_V\}$  for all  $x \in M$ .

(b)  $B_M$  is said to be whole if  $B(x) = V$  for all  $x \in M$ .

**5.2. Proposition:** Let  $A_{M_1}$  and  $B_{M_2}$  be UFSG-modules of  $V$ , then

(i) If  $A_{M_1}$  and  $B_{M_2}$  are trivial UFSG-modules of  $V$ , then  $A_{M_1} \cup_{\mathcal{R}} B_{M_2}$  is a trivial UFSG-module of  $V$ .

(ii) If  $A_{M_1}$  and  $B_{M_2}$  are whole UFSG-modules of  $V$ , then  $A_{M_1} \cup_{\mathcal{R}} B_{M_2}$  is a whole UFSG-module of  $V$ .

(iii) If  $A_{M_1}$  is a trivial UFSG-module of  $V$  and  $B_{M_2}$  is a whole UFSG-modules of  $V$ , then  $A_{M_1} \cup_{\mathcal{R}} B_{M_2}$  is a whole UFSG-module of  $V$ .

**Proof:** The proof is easily seen by definition 2.7, definition 5.2, and theorem 5.1.

**6. Application of UFSG-modules**

In this section, first we obtain the relation between IFSG-modules and UFSG-modules and then give the applications of soft pre-image, soft anti-image, lower  $\alpha$ -inclusion of soft sets and linear transformation of vector spaces on vector spaces with respect to UFSG-modules.

**6.1. Theorem:** Let  $T_M$  be a fuzzy soft set over  $V$ . Then,  $T_M$  is an UFSG-module of  $V$  if and only if  $T_M^r$  is an IFSG-modules of  $V$ .

**Proof:** Let  $T_M$  be a UFSG-modules of  $V$ . Then for all  $x,y \in M$ ,  $a,b, \alpha \in F$ ,

$$T_M^r(ax+by) = \bigvee_{T_M(ax+by)} \supseteq \bigvee_{(T_M(x) \cup T_M(y))}$$

$$= \left( \bigvee_{T(x)} \right) \cap \left( \bigvee_{T(y)} \right) = T_M^r(x) \cap T_M^r(y)$$

$$T_M^r(\alpha x) = \bigvee_{T_M(\alpha x)} \supseteq \bigvee_{T_M(x)} = T_M^r(x)$$

Thus,  $T_M^r$  is an IFSG-module of  $V$ .

Conversely, let  $T_M^r$  be an IFSG-module of  $V$ . Then for all  $x,y \in M$ , and  $\alpha \in F$ ,

$$T_M(ax+by) = \bigvee_{T_M^r(x+y)} \subseteq \bigvee_{(T^r(x) \cap T^r(y))}$$

$$= \left( \bigvee_{T^r(x)} \right) \cup \left( \bigvee_{T^r(y)} \right) = T(x) \cup T(y)$$

$$T_M(\alpha x) = \bigvee_{T_M^r(\alpha x)} \subseteq \bigvee_{T^r(\alpha x)} = T(x)$$

Thus,  $T_M$  is an UFSG-module of  $V$ .

The above theorem shows that if a fuzzy soft set is a UFSG-module of  $V$ , then its relative complement is an IFSG-module of  $V$  and vice versa.

**6.2. Theorem:** If  $T_M \lesssim_u V$ , then  $M_T = \{x \in M / T(x) = T(0_V)\}$  is a G- module of M.

**Proof:** It is seen that  $0_V \in M_T$  and  $\emptyset \neq M_T \subseteq M$ . We need to show that  $ax+by \in M_T$  and  $\alpha x \in M_T$ , for all  $x, y \in M_T$  and  $\alpha \in F$ . Since  $x, y \in M_T$  and  $T_M$  is a UFSG- module of V, then  $T(x)=T(y)=T(0_V)$ ,  $T(ax+by) \subseteq T(x) \cup T(y) \subseteq T(0_V)$ .  $T(\alpha x) \subseteq T(x) = T(0_V)$  for all  $x, y \in M_T$  and  $\alpha \in F$ . Furthermore, by proposition 5.1,  $T(0_V) \subseteq T(ax+by)$  and  $T(0_V) \subseteq T(\alpha x)$ . Thus, the proof is completed.

**6.3. Theorem:** Let  $G_M$  be a fuzzy soft set over V and  $\alpha$  be a subset of V such that  $\alpha \supseteq G(0_V)$ .

If  $G_M$  is a UFSG-module of V, then  $G_M^{\subseteq \alpha}$  is a G-module of V.

**Proof:** Since  $\alpha \supseteq G(0_V)$ , then  $0_V \in G_M^{\subseteq \alpha}$  and  $\emptyset \neq G_M^{\subseteq \alpha} \subseteq V$ .

Let  $x, y \in G_M^{\subseteq \alpha}$  and  $a, b, \alpha \in F$ , then  $G(x) \subseteq \alpha$  and  $G(y) \subseteq \alpha$ .

We need to show that  $ax+by \in G_M^{\subseteq \alpha}$  and  $\alpha x \in G_M^{\subseteq \alpha}$  for all  $x, y \in G_M^{\subseteq \alpha}$  and  $\alpha \in F$ . Since  $G_M$  is a UFSG –module of V, it follows that  $G(ax+by) \subseteq G(x) \cup G(y) \subseteq \alpha \cup \alpha = \alpha$

Therefore,  $G(\alpha x) \subseteq G(x) \subseteq \alpha$ , which completes the proof.

#### 6.4. Theorem

Let  $G_M$  and  $T_W$  be a fuzzy soft set over V, where M and W are G-modules of V and  $\Psi$  be a linear transformation from M to W. If  $T_W$  is a UFSG-module of V, then so is  $\Psi^{-1}(T_W)$ .

**Proof:** Let  $T_W$  be a UFSG-module of V. Then  $T_W^r$  is an IFSG- module of V by theorem:6.1 and  $\Psi^{-1}(T_W^r)$  is an IFSG- module of V by theorem 4.4. Thus,  $\Psi^{-1}(T_W^r) = (\Psi^{-1}(T_W))^r$  is an IFSG- module of V by theorem 2.1. Therefore,  $\Psi^{-1}(T_W)$  is a UFSG –module of V by theorem 6.1.

#### 6.5. Theorem

Let  $G_M$  and  $T_W$  be a fuzzy soft sets over V, where M and W are G- Modules of V and  $\Psi$  be a linear isomorphism from M to W. If  $G_M$  is a UFSG- Module of V, then so is  $\Psi^*(G_M)$

**Proof:** Let  $G_M$  be a UFSG- module of V. Then  $G_M^r$  is an IFSG- module of V by theorem:6.1 and  $\Psi^{(G_M^r)}$  is an IFSG- module of V by theorem 4.3. Thus,  $\Psi^{(G_M^r)} = (\Psi^*(G_M))^r$  is an IFSG- module of V by theorem 2.1. So,  $\Psi^*(G_M)$  is a UFSG- module of V by theorem 6.1.

#### 6.6. Theorem

Let  $V_1$  and  $V_2$  be two vector spaces and  $(T_1, M_1) \lesssim_u V_1$ ,  $(T_2, M_2) \lesssim_u V_2$ .

If  $f: M_1 \rightarrow M_2$  is a linear transformation of vector spaces, then

- (i) If  $f$  is surjective, then  $(T_1, f^{-1}(M_2)) \lesssim_u V_1$ ,
- (ii)  $(T_2, f(M_1)) \lesssim_u V_2$ ,
- (iii)  $(T_1, \ker f) \lesssim_u V_1$ .

**Proof:** Follows from definition 5.1 and theorem 4.5. Therefore omitted.

#### 6.1. Corollary

Let  $(T_1, M_1) \lesssim_u V_1$ ,  $(T_2, M_2) \lesssim_u V_2$ . If  $f: M_1 \rightarrow M_2$  is a linear transformation, then  $(T_2, \{0_{M_2}\}) \lesssim_u V_2$ .

**Proof:** Follows from definition 6. 6(ii) and theorem 6. 6 (iii).

**Conclusion:** Throughout this paper, we have dealt with IFSG-modules and UFSG-modules of a vector space. We have investigated their related properties with respect to soft set operations obtained the relations between IFSG-modules and UFSG-modules. Furthermore, we have derived some applications of IFSG-modules and UFSG-modules with respect to soft image, soft pre image, soft anti image,  $\alpha$ -inclusion of soft sets and linear transformations of vector spaces. Further study could be done for fuzzy soft sub structures of different algebras.

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