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Some analogous results for ring of operators on Hilbert space

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Abstract
In this research paper we explained and see that the results for ring of operators on Hilbert space is same as the ring, whenever we suppose H be a Hilbert space and R be a set of operators on H given by $R=\{Ti:Ti2=O, TiTj=O \text{ for all } i,j\}$ then (R,+) is an additive abelian group and (R,.) is multiplicative group as well as the algebraic structure (R, +,.) forms a ring. Also a special case achieved that If (R,+,.) is a Boolean ring then it is commutative ring. Also we got a beautiful result as (R,+,.) is commutative if and only if (Ti + Tj)2 = Ti2 + 2TiTj + Tj2. We got result on homomorphism and isomorphism also as Let R and R1 be rings of operators on a Hilbert space H and f: R→R1 be a mapping defined by f(Ti)=Ti, then f is Homomorphism as well as Isomorphism. Thus here we have seen some important results of the algebraic structure Ring is the same of the Ring of operators on a Hilbert Space.

Keywords: Homomorphism, operator, ring, space, hilbert space

1. Introduction
In this research paper we have constructed a set of operators R on a Hilbert space H and we have shown that R is an additive abelian group as well as multiplicative abelian group. Extending this study to the theory of ring we have also shown that R is a ring also with the operations addition and multiplication. In order to widen the range of study we have also studied some of the results for ring and we have found that these results are also true for this set R also. We in this paper, did not halt ourselves here but we travelled a bit more and studied also ring homomorphism and isomorphism and established some of the analogous results for the ring with the concept of the ring of operators. During this study out of which we have established some of the analogous results we have earned the experience to observed that the method of proving the results the method does change even a bit.

2. Preliminaries and definitions
In this section, we refer to some preliminaries on Ring, Operators and Hilbert Spaces, however we give below some of the relevant definitions to serve as a ready reference.

2.1 Linear Space: The symbol K will stand for either the set R of all real numbers or the set C of all complex numbers.
A structure of linear space on a set E is defined by two maps.
a. (x, y) → x+ y of E×E into E and is called vector addition.
b. (a, x) → ax of K×E into E and is called scalar multiplication.

The above two maps are assumed to satisfy the following conditions.
1. x + y = y + x, for every x, y in E.
2. x + (y + z) = (x + y) + z for every x, y, z in E.
3. There exists an element 0 in E such that x+0 = 0+x = x, for every x in E.
4. For every element x in E there exists an element denoted by -x in E, such that x + (-x) = (-x)+ x = 0, for every x in E.
5. A (x + y) = ax + ay, for every a in K and all x, y in E.
6. (a + b) x = ax + bx, for every a, b in K and all x in E.
7. (ab) x = a (bx), for every a, b in K and all x in E.
8. 1·x = x, for every x in E.

Whenever all the above axioms are satisfied, we say that E is a linear space (or a vector space) over K.
Now if K be the set of all real numbers then E is a real linear space and similarly if K be the set of all complex numbers then E is called a complex linear space.
Here every element of E is called a vector and every element in K is called a scalar.
The zero vector 'O' is unique and called the zero element or the origin.

2.2 Linear Subspace
Let E be a linear space (over a field K). A non-empty subset F of E is called a linear subspace (or simply subspace) of E if F itself forms a vector space over K with respect to the addition and scalar multiplication defined on E.

2.3 Zero Space
A linear space may consists solely of the vector **0** with scalar multiplication defined by $\alpha\cdot 0 = 0$, for every α. We call this linear space as zero space and we always denote it by {0}.

2.4 Linear Transformation
Let E and E′ be any two linear spaces (over the field K). A mapping $T:E\rightarrow E'$ is called a linear transformation if the following conditions are satisfied.
(i) T (u+v) = T (u)+T (v), for every u, v are in E. (ii) T(αu) = αT (u) for every u∈E, and α is in K. Here the conditions (i) and (ii) can be together expressed as, T (αu+βv) = αT (u) +βT (v), for every u, v in E and α, β in K.
A linear transformation is also called a linear mapping. If there be no chance of confusion then we shall write T_x in place T(x). Now it is easy to see that, If $T:E\rightarrow E'$ be a linear transformation of a linear space E in to a linear space E′ then T preserves the origin and negatives. Since $T(0)=T(0\cdot 0)=0\cdot T(0)=0$. Also, $T(-x)=T(-1\cdot x)=(-1)\cdot T(x)=-T(x)$, for every x in E.

2.5 Linear Operator
Let E be a linear space (over the field K), then a mapping T: E→E′ from a linear space E in to E itself is called a linear operator on E, if it satisfies the following conditions. T (αu+βv)=αT(u)+βT (v), for every u,v in E and α,β in K.
Thus T is a linear operator on a linear space E if T is a linear transformation from E in to E itself.

2.6 Zero Transformation
Let E and E′ be any two linear spaces (over the field K). Let T:E→E′ be a mapping from E in to E′. Now, if T is defined as T(u)=0 (zero vector of E′) for every u in E, then for u, v ∈ E; α, β ∈ K. we have au+βv ∈ E. But then, $T(\alpha u+\beta v)=0 = \alpha\cdot 0 + \beta\cdot 0 = \alpha T(u)+\beta T(v)$
Hence, T in this situation is a linear transformation from E in to E′ and T is called a zero transformation. If there be no chance of confusion then whenever T is a zero transformation we shall denote it by 0 (zero).

2.7 Negative of A Linear Transformation
Let E and E′ be any two linear spaces (over the field K). Let T:E→E′ be a linear transformation from E into E′. Then the negative of T denoted by –T is defined by (-T) (u)=- [T(u)] for every u ∈ E and T(u) ∈ E′. Since, T (u) ∈ E′ ⇒ -T(u) ∈ E′. Thus, –T is also a function from E into E′.

Now, let α,β ∈K and u, v∈E. Then, αu+βv ∈ E Also, $(-T)(\alpha u+\beta v) = -[T(\alpha u+\beta v)] = -[\alpha T(u)+\beta T(v)]=\alpha[-T(u)]+\beta[-T(v)]$ Thus, –T is a linear transformation from E in to E′.[since T is a linear transformation]
Hence, –T is called the negative of the linear transformation T.

2.8 The inverse of a Linear Transformation
Let E and E′ be any two linear spaces (over the field K). Let T: E→E′ be a linear transformation from E in to E′.
We shall say T is one-one if, for $x_1,x_2 \in E$ and $x_1 \neq x_2 \Rightarrow T(x_1) \neq T(x_2)$
That is, T is one-one if, for $x_1,x_2 \in E$ and $T(x_1) = T(x_2) \Rightarrow x_1 = x_2$
Further, T is onto if, for y ∈ E′ ⇒ ∃ x in E, such that T(x) = y.
When T is onto and one-one then a function of the form of $T^{-1}: E'\rightarrow E$, a function from E′ in to E, is read as the inverse of T and we define it as, Let y∈E′ be an arbitrary vector in E′.Since T is onto so, y ∈ E′ ⇒ ∃ x in E, such that T(x)=y. Also, x determined in this way is unique element of E because T is one-one.
Thus, $x_1,x \in E$ and $x_1 \neq x \Rightarrow y = T(x) \neq T(x_1)$. Then we define $T^{-1}(y) = x$. Thus we can say that, $T^{-1}:E'\rightarrow E$ such that $T^{-1}(y) = x \Leftrightarrow T(x) = y$. The function T^{-1} is itself one-one and onto.

2.9 Range of a Linear Transformation
Let T:E→E′ be a linear transformation from a linear space E into a linear space E′.Then range of T denoted by R(T) or equivalently as RT is the set of all vectors y in E′ such that, y = T(x), for same x in E. Thus the range of T is the image set of E under T. Thus, Range (T) = {T(x) ∈E′ for x in E}.

2.10 Null Space of a Linear Transformation
Let T:E→E′ be a linear transformation from a linear space E into a linear space E′. Then the null space of T will be denoted by N(T) or equivalently by N_T is defined as the set of all vectors x in E, such that, T(x)=0 (where 0 is the zero vector of E′). Thus, N (T) ={x ∈ E: T(x) = 0 ∈ E′}.

2.11 Kernel of Linear Transformation: Let T:E→E′ be a linear transformation from a linear space E into a linear space E′, then the 'Kernel of T'is denoted by 'KerT' and is defined by KerT={x∈E:T(x)=0∈E′ (the zero vector of E′)}.

2.12 Identity Operator
Let E be a linear space (over the field K). Let I:E→E be a mapping from the linear space E into E itself. Now let I be defined as I (u) = u, for every u ∈E.
Then for, α, β ∈ K; u, v ∈ E. αu + βv ∈ E. And then, I (αu+βv)= αu+βv = αI(u)+βI(v).
Thus, I is a linear transformation from E into E itself and we say, I the identity operator on E.

2.13 Kernel of an Identity Transformation
Let I:E→E be an identity transformation then 'Ker of I' denoted by 'KerI' is defined as, KerI={x ∈E:I(x)=x=0}={0}.

2.14 Kernel of the Zero Transformation
Let 0:E→E be a zero transformation from a linear space E in to E itself. Then the "Kernel 0" is denoted by 'Ker 0' and is defined as, Ker 0 ={x ∈ E,: 0(x)=0}=E.

2.15 The Sum of Two Linear Transformations
Let T_1, T_2 be two linear transformations from linear space E into linear space E′ then T_1+T_2 is also a linear transformation from E into E′.
(We refer to Jha, K.K (1) page 168-169).
As on the same ground the sum of a finite number of linear transformations is again a linear transformation.
Similarly αT_1 is also a linear transformation.

2.16 The Product of Two Linear Transformations
For any two linear transformations T_1, T_2 on E we define $(T_1T_2)(x)=T_1(T_2(x))$, x∈ E. We call T_1T_2 as the product of T_1 and T_2.
The product T_1T_2 is also a linear transformation because $(T_1T_2)(\alpha x+\beta y) = T_1(T_2(\alpha x + \beta y)) = T_1(\alpha T_2(x) + \beta T_2(y)) = \alpha T_1(T_2(x)) + \beta T_1(T_2(y)) = \alpha(T_1T_2)(x) + \beta(T_1T_2)(y)$, ∀ α,β are scalars, x,y ∊ E.
(We refer to Jha, k.k (1) page 169)

2.17 Non Singular Linear Transformation
Let T be a linear transformation on a linear space E then T is called invertible or nonsingular if T is one-one and onto otherwise T is called singular. Thus if T is a non-singular linear transformation on a linear space E then the transformation T:E→E is one-one and onto. Hence its inverse, T^{-1}:E→E exists such that, $T(x) = y \Leftrightarrow x = T^{-1}(y)$ Also, if I is the identity function on E. Then, $TT^{-1} = I = T^{-1}T$ Further, (i) T^{-1} is also a linear transformation on E, (ii) A linear transformation on a finite dimensional vector space is E is non-singular if and only if it is one to one. (For proof of (i) and (ii) we refer to Jha, K.K (1) p-172) (iii) A linear transformation T on a linear space E is one-one if and only if T is onto.

2.18 Norm
Let E be a linear space over a field K. Then by a norm on E we understand a map $f:E \to R^+$ from E into R^+ which satisfied the following conditions.

- F (x) = 0 ⇔ x = 0.
- F (λx) = |λ| f(x) ∀ λ ∊ K, x ∊ E.
- F (x + y) ≤ f(x) + f(y) ∀ x,y ∊ E.

Now if there be no chance of confusion in writing || || for f and ||x|| for f(x) then the all above three conditions take the form.

- ||x|| = 0 ⇔ x = 0.
- ||λx|| = |λ| ||x|| ∀ λ ∊ K, x ∊ E.
- ||x+y|| ≤ ||x|| + ||y|| ∀ x,y ∊ E.

(Where R^+ denotes the set of non-negative real numbers)

2.19 Normed Linear Space
By a normed linear space we understand a linear space E together with a norm || || defined on it. At times a normed linear space is also called a normed vector space or a normed space.

2.20 Metric (or Distance Function)
Let M be a non-empty set then a real valued function d defined on M×M is called a distance function (or metric function or simply metric on M) if the following conditions are satisfied

- d (x, y) ≥ 0.
- d (x, y) = 0 ⇔ x = y.
- d (x, y) = d(y, x).
- d (x, z) ≤ d (x, y) + d(y, z).

Here the condition (iii) is known as the condition of symmetry and condition (iv) is known as triangle inequality. Also, d (x, y) due to symmetry does not depend on the order of the elements.

2.21 Metric Space
The system (or the pair) (M, d) containing a non-empty set M and a metric d defined on it is called a metric space. The elements of M are called the points of the metric space (M, d). If there is no chance of confusion, we denote the metric space (M, d) by the symbol M which is used for the underlying set of points. One should always keep in mind that a metric space is not merely a non-empty set.
(For the definition of metric and metric space, we refer to Simmons, G.F (1) (p-51)).
Now we can verify that the normed linear space N is a metric space with respect to the metric d defined by d (x, y)= ||x-y||. (For verification we refer to Jha, k.k (1)p-181).

2.22 Cauchy Sequence
A sequence (x_n) of points of a metric space (M,d) is said to be a Cauchy sequence, if for each $\varepsilon>0$ however small, there exists a natural number n_t dependent on ε such that, m, n ≤ N and m, n ≥ n_t ⇒ $d(x_m, x_n) < \varepsilon$
That is, if $\lim_{x\to\infty} d(x_m, x_n) = 0$.
It is worth much to note that. (1) Every convergent sequence is a Cauchy sequence. (2) Cauchy sequence is not necessarily convergent. (We refer to Simmons, G.F (1) p-71).

2.23 Complete Metric Space
A metric space (M, d) is said to be complete if every Cauchy sequence in (M, d) is convergent in (M, d). (We refer to Simmons, G.F (1) p-71).

2.24 Banach Space
A normed linear space N is said to be a Banach space if it is complete as a metric space.

2.25 Inner Product (or Scalar Product)
Let E be a linear space over a field K. By an inner product (or scalar product) on E we mean a map (x, y) → (x/y) of E×E into K. and satisfying the following conditions namely.

1. (x/x) ≥ 0 ∀ x in E.
2. (x/x) = 0 ⇔ x = 0.
3. $(x/y) = (\overline{y/x})$.
4. (λx + μ y/z) = λ(x/z) + μ(y/z) ∀ λ, μ ∈ K, x, y, z ∈ E.

However if K = R (the set of real numbers), then, the condition (iii) takes the form (x/y) = (y/x).
Also, from conditions (iii) and (iv) we have $(x/\lambda y + \mu z) = (\overline{\lambda y + \mu z/x}) = \overline{\lambda(y/x) + \mu(z/x)} = \bar{\lambda}(\overline{y/x}) + \bar{\mu}(\overline{z/x}) = \bar{\lambda}(x/y) + \bar{\mu}(x/z)$, (We refer to Simmons, G.F (1) (p-245))
It is worthy much to note that an inner product space E is a normed linear space with respect to the norm defined in term of an inner product given by $\|x\| = +\sqrt{x/x}$.
(For verification we refer to Jha, K.K (1) p-270)
There is no chance of confusion then (x/y) is read as the inner product of x with y (or equivalently the dot product of x with y or the dot product of x and y.

2.26 Hilbert Space
A Banach space is said to be a Hilbert space if its norm is or can be defined by means of an inner product.

2.27 Operator on a Hilbert Space H
By an operator on a Hilbert space H we shall mean a continuous linear transformation from H into self.

2.28 Set of Operators R on A Hilbert Space H
We construct a set of operators R on a Hilbert space H in the following way: $R=\{T_i:T_i^2=0, T_iT_j=0\}$

2.29 Binary Operation
Let E be a non-empty set then by a binary operation on E we mean a mapping $O:E\times E\rightarrow E$ from E×E into E. given by, $O:(x,y)\rightarrow xOy \in E$ for all x, y ∈ E and $(x,y) \in E\times E$.That is the closure property is satisfied.

2.30 Group
Let G be a non-empty set and 'O' be a binary operation defined on G. Then the pair (G,O) is said to be a group if and only if the following conditions are satisfied: (i) As 'O' is a binary operation so the closure property is automatically satisfied. (ii) The binary operation 'O' is associative That is, (aOb) Oc=aO (bOc) for all a, b, c ∈ G. (iii) Existence of an identity elements. For every element a ∈ G ∃ an element e in G such that, aOe = eOa = a (iv) Existence of an inverse elements. For every a∈G ∃ an elements a^{-1} read as the inverse of a, such that, $aOa^{-1} = a^{-1}Oa=e$.
However if an additinal condition of commutiative is satisfied. i.e. a O b = b O a ∀ a, b∈G.
Then, (G,O) is called an Abelian group or a commutative if there be no chance of confusion then in place of writing (G,O) we simply write G.

2.31 Sub Group
A non-empty subset H of a group G is said to be a subgroup of G. If H forms a group under a binary composition of G. Obviously if H is a subgroup of G, and K is a subgroup of H. Then K is a subgroup of G. If G is a group with identity element e. Then the subsets {e} and G are trivially subgroups of G and we call them the trivial subgroups. All other subgroups will be called nontrivial (or proper subgroups). Without any hesitation we sometimes use the notation H≤G to signify that H is a subgroup of G, and H=G to mean that H is a proper subgroup of G.
It may be a little cumbersome at times to check whether a given subset H of a group G is a subgroup or not by having to check all the axioms in the definition of a group. We give here two results on subgroup which analogous result has been established in the next section with the concept of operator on a Hilbert space. (α)→ A non-empty subset H of a group G is a subgroup of G if and only if (i) a,b ∈ H ⇒ ab ∈ H (ii) a ∈ H ⇒ a^{-1} ∈ H. (β)→ (i) A non void subset H of a group G is a subgroup of G if and only if a, b ∈ H ⇒ ab^{-1} ∈ H (ii) A non empty finite subset H of a group G is a subgroup of G if and only if H is closed under multiplication. (We refer to Bhambri S.K. (1) p-63-64)

2.32 Homomorphism
Let (G,O) and (G′,O′) be two groups then a mapping $f:G\rightarrow G'$ is called a homomorphism if f(a O b) = f(a) O′ f(b). for a, b ∈ G when there is no scope of confusion we shall use the same symbol '. ' in place of 'O' and ' O′ '. Hence the above definition at once takes the form $f:G\rightarrow G'$ is a homomorphism if f(a b) = f(a) f(b).
That is f preserves the composition in groups.

2.33 Isomorphism
The function $f:G\rightarrow G'$ is an isomorphism
If (i) f preserves the composition in groups, (ii) f is one-one and (iii) f is onto.
Clearly every isomorphism is a homomorphism or an isomorphism is a special case of homomorphism.
Also if, $f:G\rightarrow G'$ is homomorphism then we say that f(G) is the homomorphic image of G in G′.
Also G′ is called homomorphic image of G if f is onto homomorphism and whenever $f:G\rightarrow G'$ is isomorphism then f(G) is called an isomorphic image of G in G′.
In this situation we also say that G is isomorphic to G′ or G is equivalent to G′.

2.34 Ring Homomorphism
Let R and R^1 be any two rings. Let $f:R\rightarrow R^1$ be a mapping from R to R' then f is called homomorphism (or ring homomorphism) if it satisfied the following conditions: (i) f(a+b)=f(a)+f(b) and (ii) f(ab)=f(a)f(b) for all a,b ∈ R.

2.35 Ring Isomorphism
A ring homomorphism is called a ring isomorphism if it is also one-one and onto. Whenever $f:R\rightarrow R^1$ is an isomorphism than R is called isomorphic to R^1, Also if $f:R\rightarrow R$ is an isomorphism then we say that f is an automorphism.

2.36 Boolean Ring
A ring is called a Boolean ring if $a^2 = a$ ∀ a ∈ R.

3. Results and Discussion
In this section we establish some of the results by making the use of the definitions given in above section.

Theorem (3, I): Let H be a Hilbert space. R be a set of operators on H given by $R=\{T_i:T_i^2=O, T_iT_j=O$ for all i,j$\}$ Then (R,+) is an additive abelian group.

Proof: Since $O^2 = O$ and $OT_j = O$ for all j
It follows that the zero operators belongs to R and here R is non empty.
That R ≠ ∅
We define additio0050*n of operators in R as usual and find that for $T_i,T_j \in R$,
$(T_i+T_j)^2 = T_i^2 + 2T_iT_j + T_j^2 = O \in R$
Thus $T_i + T_j = 0$ implies that $T_i+T_j \in R$
Hence R is closed with respect to addition.
Also since operators obey the associative law with respect to addition.
Thus for T_i, T_j and $T_k \in R$ we have $(T_i + T_j) + T_k = T_i + (T_j + T_k)$.
Also zero operator is the additive identity of R.
Since $O + T_i = T_i + O = O$ for all i.
Also $T_i \in G$ implies $T_i^2 = O$ (by the construction of the set R).
⇒ $(-T_i)^2 = O$ ⇒ $-T_i \in G$
That is for $T_i \in R$ we see that $-T_i \in R$
Also $T_i + (-T_i) = O$ (operator) the identity operator.
Also $T_i + T_j = T_j + T_i$
Thu11`11111s R is an additive abelian group.

Theorem (3, II): Set R = $\{T_i : T_i^2 = O, T_i . T_j = O$ for all i,j} of non-singular operators on a Hilbert space H, Then (R,.) is multiplicative group.

Proof: We have seen in theorem (3,I) that operator O is in R. That is $R \neq \emptyset$
Also for $T_i, T_j \in R$,
$(T_iT_j)^2 = (T_i)^2 (T_j)^2 = T_i^2 . T_j^2$
$= O.O = O \in R$
Thus for $T_i, T_j \in R$, $Ti.Tj \in R$ implies R is closed with respect to multiplication.
Also operators satisfy associative law with respect to multiplication.
Also, for $T_i \epsilon G$, T_i^{-1} exists such that T_i^{-1} is also non-singular.
Thus $T_i^{-1} \in R$
Thus we find that for T_i in R we get T_i^{-1} (the inverse of T_i) in R.
Also $T_iT_i^{-1}=T_i^{-1}T_i=I$
Hence the existence of inverse element.
Also for $T_i \in R$ we have $T_iI=IT_i=T_i$
So the identity element exists.
Also $T_iT_j=T_jT_i$
Thus, (R,) is a multiplicative abelian group.

Theorem (3, III): The system (R, +,.) is a ring.
Proof: we see that (i) $T_i,T_j \in R \Rightarrow T_iT_j = O \in R$
Thus R is closed with respect to multiplaction.
(ii)Also elements of R commute since
$T_iT_j = 0 = T_jT_i$ for all I,j
Hence $(T_i+T_j)^2=T_i^2+T_j^2+2T_iT_j=0 \epsilon R$.
Thus $T_i,T_j \epsilon R$ implies that $T_i+T_j \epsilon R$
Therefore R is closed with respect to addition
(iii) zero operator is the additive identity of R.
(iv) since for $T_i \epsilon R$ implies $T_i^2=0 \Rightarrow (-T_i)^2 = T_i^2 = 0 \Rightarrow -T_i \epsilon R$
Hence the additive inverse $–T_i$ of T_i is also in R.
Finally. Since elements of Rare operators, and associative as well as commutative laws of addition. Associative law of multiplication and distributive laws hold good for operator, these laws also hold good for elements of R.
Hence R is ring.
Remark: - In a ring (R,+,.) show that $(T_i+T_j)^2 =T_i^2 +T_iT_j+ T_jT_i+T_j^2$, Where $T_i,T_j \epsilon R$.

Proof: Since $(T_i+T_i)^2 = (Ti+T_j)(T_i+T_j) = (T_i+T_j)T_i + (T_i+T_j)T_j = T_i^2+T_jT_i+T_iT_j+T_j^2$

Theorem (2.3, IV): If (R,+,.) is a boolean ring then it is commutative.

Proof: Since $T_i^2=T_i$
This implies that $(T_i+T_j)=(T_i+T_j)^2=(T_i+T_j)(T_i+T_j)$

$= (T_i+T_j) T_i+(T_i+T_j)T_j$

$=T_i^2+T_jT_i+T_iT_j+T_j^2$

$=T_i+T_jT_i+T_iT_j+T_j$ [since $T_i^2=T_i$]

Implies that $T_i+T_j=T_i+T_jT_i+T_iT_j+T_j$

Implies that $Ti+Tj-T_i-T_j=T_jT_i+T_iT_j$

$\Rightarrow T_jT_i+T_iT_j=0$

$\Rightarrow T_iT_j=-T_jT_i$ (2.11)

Also $T_i^2=T_i$ (2.12)

$(-T_i)^2=-T_i$

That is $(-T_i)(-T_i)=-T_i$

$\Rightarrow T_i^2=-T_i$ (2.13)

Thus from [(2.12) and (2.13)]

$T_i=-T_i$

Then keeping this value in (2.11) we have,

$-T_iT_j=-T_jT_i$

Hence $T_iT_j=T_jT_i$

Therefore R is commutative.

Also $T_i=T_i^2=0 \Rightarrow 2T_i=0$.

Theorem (2.3, V): Ring (R,+,.) is commutative if and only if $(T_i+T_j)^2=T_i^2+2T_iT_j+T_j^2$
Proof: Let us assume that R is commutative
Then $(T_i+T_j)^2=(T_i+T_j)(T_i+T_j)=T_i^2+T_iT_j+T_jT_i+T_j^2$
But R is commutative so $T_iT_j=T_jT_i$
Thus $(T_i+T_j)^2=T_i^2+2T_iT_j+T_j^2$
Conversely, Let $(T_i+T_j)^2=T_i^2+2T_iT_j+T_j^2$ for all $T_i,T_j \epsilon R$.
Then $(T_j+T_i)^2=T_j^2+2T_jT_i+T_i^2$
Since the ring R is additive abelian group
Hence $T_i+T_j=T_j+T_i$
Thus $(T_i+T_j)^2=(T_j+T_i)^2$
Implies that $T_i^2+2T_iT_j+T_j^2=T_j^2+2T_jT_i+T_i^2$
$\Rightarrow 2T_iT_j=2T_jT_i$
Hence $T_iT_j=T_jT_i$
$\Rightarrow$R is commutative.

Theorem (2.3, VI): (i) Let R and R' be rings of operators on a Hilbert space H. (ii) f: R→R' be a mapping defined by f $(T_i)=T_i$, Then f is Homomorphism as well as isomorphism.
Proof: since f $(T_i+T_j)=T_i+T_j=f(T_i)+f(T_j)$
Also f $(T_iT_j)=TiTj=f(T_i)f(T_j)$
Thus f is homomorphism.
Also f $(T_i)=f(T_j) \Rightarrow T_i=T_j$
Thus f is one-one.
Again, for each T_i in R we get a $f(T_i)$ in R' and each $f(T_i)$ in R' is associated with $T_i \epsilon R$.
Therefore f is onto.
Thus we find that f is homomorphism, one-one and onto.
Thus f is isomorphism.

Theorem (2.3, VII): Let R and R' be two rings of the operator on a Hilbert space H and f:R→R' be an isomorphism of ring R onto ring R'
Then R is commutative ring if and only if R' is commutative ring.

Proof: Let R be a commutative ring
Then we have already seen in[theorem(2.3,XI)] that R' is commutative even if f is homomorphism in place of isomomorphism.

So in this case also R' will be commutative.
Conversely we suppose that R' is commutative then it Remains to show that R is also commutative.
For this, since f is isomorphism, f is one-one, onto and homomorphism.
Hence f^{-1} exists and is one-one, onto.
Also for $T_i, T_j \epsilon R$ there exists T_i', T_j' in R' such that

$F(T_i)=T_i' \Rightarrow f^{-1}(T_i')=T_i$

$F(T_j)=T_j' \Rightarrow f^{-1}(T_j')=T_j$

Now $T_iT_j=f^{-1}(T_i')f^{-1}(T_j')=f^{-1}(T_i'T_j')$

$= f^{-1}(T_j'T_i')$

$=f^{-1}(T_j').f^{-1}(T_i')$

$= T_jT_i$

That $T_iT_j=T_jT_i$ for T_i,T_j are in R

Therefore R is also commutative.
In fact there remains still a vast scope to study it to establish analogous results for ring. so we left our study here for the students of the school of mathematics who desire to develop this study further.

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