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Study of Riemann-Liouville Weyl approaches in fractional calculus

Kamlesh Kumar Saini

Abstract
In this paper introduces fractional derivatives and fractional integrals. After a short introduction and some preliminaries the Riemann-Liouville and weyl approaches in fractional calculus will be explored. Then some basic properties of Riemann-Liouville and weyl approaches, such as the Cauchy formula for integration, namely, semi group property of fractional differintegral operators, There after the definitions of the Riemann-Liouville and weyl approaches will be applied to a few examples. Also fractional differential equations and one method for solving them will be discussed. The paper ends with some examples of Riemann-Liouville and weyl approaches in fractional calculus.

Keywords: Fractional calculus, fractional derivative, Riemann-Liouville and weyl approaches

Introduction

A. Gamma Function: The gamma function is a generalization of factorials of real numbers [1]. It can be defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \qquad x \in R^+$$

The Gamma function has the following properties

$$\Gamma(x+1) = x\Gamma(x) \qquad x \in R^+$$
$$\Gamma(x) = (x-1)! \qquad x \in N$$

The gamma function is utilized in a wide range of applications in Signal Processing, such as Fourier domain analysis, Filter design, Wavelet transformation, etc.

B. Beta Function:- The beta function is a generalized form of the definite integral [1]. It can be defined as

$$\beta(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt \qquad x, y \in x \in R^+$$

Beta function in terms of Gamma function

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \; x, y \in x \in R^+$$

C. Introduction to Fractional calculus:

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number powers or complex number powers of the differential operator

$$D = \frac{d}{dx}$$

and the integration operator J. (Usually J is used instead of I to avoid confusion with other I-like glyphs and identities.)
In this context the term powers refers to iterative application or composition, in the same sense that –

$f^2(x) = f(f(x))$

For example, one may ask the question of meaningfully interpreting

$\sqrt{D} = D^{1/2}$

as a square root of the differentiation operator (an operator half iterate), i.e., an expression for some operator that when applied twice to a function will have the same effect as differentiation. More generally, one can look at the question of defining

D^a

for real-number values of a in such a way that when a takes an integer value n, the usual power of n-fold differentiation is recovered for n > 0, and the $-n^{th}$ power of J when n < 0.
There are various reasons for looking at this question. One is that in this way the semi group of powers D^n in the discrete variable n is seen inside a continuous semi group (one hopes) with parameter a which is a real number. Continuous semi groups are prevalent in mathematics, and have an interesting theory. Notice here that fraction is then a misnomer for the exponent, since it need not be rational, but the term fractional calculus has become traditional. Fractional differential equations are a generalization of differential equations through the application of fractional calculus.

D. Fractional derivative

An important point is that the fractional derivative at a point x is a local property only when a is an integer; in non-integer cases we cannot say that the fractional derivative at x of a function f depends only on the group of f very near x, in the way that integer power derivatives certainly do. Therefore it is expected that the theory involves some sort of boundary conditions, involving information on the function further out. To use a metaphor, the fractional derivative requires some peripheral vision.
As far as the existence of such a theory is concerned, the foundations of the subject were laid by Liouville in a paper from 1832. The fractional derivative of a function to order a is often now defined by means of the Fourier or Mellin integral transforms.

E. Heuristics

A fairly natural question to ask is whether there exists an operator H, or half-derivative, such that

$H^2 f(x)= Df(x)= \frac{d}{dx} f(x)= f'(x)$

It turns out that there is such an operator, and indeed for any a > 0,

there exists an operator P such that

$(p^a f)(x) = f'(x)$

or to put it another way, the definition of $\frac{d^n y}{dx^n}$ can be extended to all real values of n. to delve into a little detail, start with the Gamma function Γ, which extends factorials to non integer values. This is defined such that

$n! = \Gamma[(n+1)$

Assuming a function f(x) that is defined where x > 0, form the definite integral from 0 to x. Call this

$(Jf(x) = \int_0^x f(t)\, dt$

Repeating this process gives

$(J^2 f)(x) = \int_0^x f(t)\, dt = (\int_0^t f(s) ds)dt$

and this can be extended arbitrarily. The Cauchy formula for repeated integration, namely

$(J^n f)(x)=\frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt$

leads to a straight forward way to a generalization for real n. Simply using the Gamma function to remove the discrete nature of the factorial function (recalling that Γ(n+1)=n!or equivalently Γ(n)=(n−1)!) gives us a natural candidate for fractional applications of the integral operator.

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt$$

This is in fact a well-defined operator.
It can be shown that the J operator satisfies

$$(J^\alpha)(J^\beta)f = (J^\beta)(J^\alpha)f = (J^{\alpha+\beta})f = \frac{1}{\Gamma(\alpha+\beta)}\int_0^x (x-t)^{\alpha+\beta-1} f(t)dt$$

this relationship is called the semi group property of fractional differintegral operators. Unfortunately the comparable process for the derivative operator D is significantly more complex, but it can be shown that D is neither commutative, nor additive in general.

Riemann - Liouville and Weyl approaches:
Let us consider some of the starting points for a discussion of classical fractional calculus; we will also introduce notations. One development begins with a generalization of repeated integration. Thus if f is locally integrable on (a,∞), then the n-fold iterated integral is given by

$$aI_x^n f(x) = \int_a^x du_1 \int_a^{u_1} du_{2\ldots} \int_a^{u_{m-1}} (f(u_n)du_n$$

$$=\frac{1}{(n-1)!}\int_a^x (x-u)^{n-1} f(u)du \quad \ldots(1)$$

for almost all x with $-\infty<a<x<\infty$ and n ∈ N. Writing $(n-1)!=\Gamma(n)$, an immediate generalization is the integral of f of fraction order $\alpha > 0$.

$aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-u)^{\alpha-1} f(u)du$ (right hand) ...(2)

and similarly for -∞ <x<b<∞

$xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)_x}\int_x^b (u-x)^{\alpha-1} f(u)du$ (left hand) ...(3)

both being defined for suitable f. The subscripts in I denote the terminals of integration (in the given order). Note the kernel $(u-x)^{\alpha-1}$ or (3). Observe that (2) for $\alpha = n$ can be shown to be the unique solution of the initial value problem

$y^{(n)}(x) = f(x), y(a) = y^1(a) = \cdots = y^{(n-1)}(a) = 0$ ...(4)

When $a = \infty$ equation (2) is equivalent to Liouville's definition, and when $a = 0$ we have Riemann's definition (without the complementary function). One generally speaks of $aI_x^\alpha f$ as the Riemann-Liouville fractional integral of order α of f, a terminology introduced by Holmgren (1863-64). On the other hand, one usually refers to

$xW_\infty^\alpha f(x) = xI_\infty^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (u-x)^{\alpha-1} f(u)du$ .....(5)

$-\infty W_\infty^\alpha f(x) = -\infty I_\infty^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-u)^{\alpha-1} f(u)du$ ...(6)

As Weyl frctional integrals of order α, they being defined for suitable f. The right and left hand fractional integrals $aI_x^\alpha f(x)$ and $xI_b^\alpha f(x)$ are related via the Parseval equality (fractional integration by parts) which we give for convenience for $a = 0$ and $b = \infty$.

$\int_0^\infty f(x)(\mathrm{o}I_x^\alpha g)(x)dx = \int_0^\infty (xW_\infty^\alpha f)(x)g(x)dx$ ...(7)

The following properties are stated for right handed fractional integrals (with obvious changes in the case of left handed integrals).

Concerning existence of fractional integrals, let $f \in L_{loc}^l(a,\infty)$. Then, if a > -∞, $aI_x^\alpha f(x)$ is finite almost everywhere on (a,∞) and belong to $L_{loc}^l(a,\infty)$. If $a = -\infty$, it is assumed that f behaves at $-\infty$ such that the integral (2) converges. Under these assumptions the fractional integrals satisfy the additive index law (or semi group property)

$aI_x^\alpha I_x^\beta f = aI_x^{\alpha+\beta} f \ (\alpha,\beta > 0)$ ...(8)

Indeed, by Dirichlet's formula concerning the change of the order of integration we have –

$$aI_x^\alpha aI_x^\beta f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-u)^{\alpha-1} du \frac{1}{\Gamma(\beta)}\int_a^u (u-t)^{\beta-1}) f(t)dt$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(t)dt \int_t^x (x-u)^{\alpha-1}(u-t)^{\beta-1}du \quad ...(9)$$

The second integral on the right equals, under the substitution y $y = \frac{u-t}{x-t}$.

$$(x-t)^{\alpha+\beta-1}\int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy = B(\alpha,\beta)(x-t)^{\alpha+\beta-1}$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-t)^{\alpha+\beta-1} \quad ...(10)$$

$B(\alpha,\beta)$ being the Beta- function. When this is substituted into the above the result follows. In particular, we have

$aI_x^{n+\alpha} f = aI_x^n aI_x^\alpha f (n \in N, \alpha > 0)$ ...(11)

which implies by n-fold differentiation for almost all x.

$$\frac{d^n}{dx^n} aI_x^{n+\alpha} f(x)(n \in N, \alpha > 0)$$

The above results also hold for complex parameters α, if the condition α > 0 is replaced by α > 0. Then the operation aI_x^α may be considered as a holomorphic function of α for $Re\alpha > 0$ which can be extended to the whole complex plane by analytic continuation, if f is sufficiently smooth.

To understand and establish this fact, we assume, for convenience, that f is an infinitely differentiable function defined on R with compact suppport contained in $[a,\infty)$ if a>-∞, implying that $f^{(n)}$(a) = 0 for n = 0, 1, 2, ...Then for any fixed x>a the integral in (8) is a holomorphic function of α for α >0. Now, integration by parts n-times yields.

$aI_x^\alpha f(x) = aI_x^{n+\alpha} f^{(n)}(x)$ (Re α> 0, n ∈ N) ...... (12)

Applying the semi group property (3.4.11) to the expression on the right in (12) and differentiating the result n-times with respect to x, we obtain.

$\frac{d^n}{dx^n} aI_x^\alpha f(x) = aI_x^\alpha f_{(x)}^{(n)}$ (Re $\alpha > 0$, n ∈ N) .... (13)

Showing that under the hypotheses assumed on f the operations of integration of fractional order α and differentiation of integral order n commute.

Conclusions

This paper introduced the concept of Fractional Calculus and Riemann-Liouville and weyl approaches in fractional calculus, the branch of Mathematics which explores fractional integrals and derivatives. We first gave some basic techniques and functions, such as the Gamma function, the Beta function and the fractional derivative. which were necessary to understand the rest of this paper. Thereafter we proved the construction of the Riemann-Liouville and weyl approaches in fractional calculus Therefore we used the fractional derivative and Cauchy formula for repeated integration respectively. Next we studied Fractional Linear Differential Equations. Riemann-Liouville and weyl approaches by the Dirichlet's formulas. I also think that the formulas are pretty awkward, definitely for first year students. It would be a lot harder to compute just a simple integer order derivative or integral. Though it is a very interesting subject and definitely worth researching, I believe it should be left as an 'exotic' branch of Mathematics.

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