International Journal of Applied Research 2017; 3(6): 1480-1483

ISSN Print: 2394-7500
ISSN Online: 2394-5869
Impact Factor: 5.2
IJAR 2017; 3(6): 1480-1483
www.allresearchjournal.com
Received: 15-05-2017
Accepted: 23-06-2017

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On a certain problem in geometric probability concerning regular polygons

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Abstract
In this paper, we consider the region inside a regular polygon for which every point is nearer to the centre of the polygon than to any of the bounding sides and compute the probability of a random point falling inside the region.

Keywords: Polygon, geometric probability, asymptotic

Introduction

Geometric probability is a branch of mathematics concerned with the study of random spatial models generated from geometric figures in Euclidean space, including classical objects such as points, lines, subspaces, balls, and convex polytopes [1]. The origins of geometric probability can be traced back to Buffon's pioneering investigation of the falling needle in 1733, which was followed by several other classic problems, including Sylvester's four-point problem in 1864 and Bertrand's paradox related to a random chord in the circle in 1888 [3, 5]. These questions were initially regarded as recreational mathematics due to the lack of a theoretical foundation, which may explain the apparent paradoxes associated with Bertrand's problem. However, the introduction of integral geometry by W. Blaschke in the early twentieth century provided a solid theoretical framework for the study of geometric probability [4]. In particular, the course given by G. Herglotz in 1933 paved the way for the development of integral geometry, which has since become an important area of research in mathematics. Integral geometry provides powerful tools for analysing random spatial models and has many applications in real-world problems, including those in physics, biology, and computer science [2]. In this paper, we focus on the problem of finding the probability of a point falling in a region inside a polygon such that every point in the region is nearer to the center than to any of the bounding sides. This problem can have potential applications in various fields, such as image processing, computer graphics, and spatial statistics.

Statement of the Problem

Given a regular polygon of n number of sides, we study the region the region in which every point is nearer to the center than to any of the bounding sides, where center is taken to be the center of the circumscribed circle. If T_n is a regular polygon of given side length with n sides we undertake to study statistically and computationally the region D as below.

$$D_n = \{(x, y) : d\big((x, y), (x_0, y_0)\big) < p\} \qquad (1)$$

Where,
(x, y): $x, y \in T_n$ and,
$p = \min p_i(x, y)$: $1 \leq i \leq n$, $p_i(x, y)$ being the distance of (x, y) from the ith side.

Solution

Without loss of generality, let one of the sides OP, of the polygon with unit side length be along x-axis so that

$O = (0,0)$, and $P = (1,0)$,

From the Figure 1, we can see

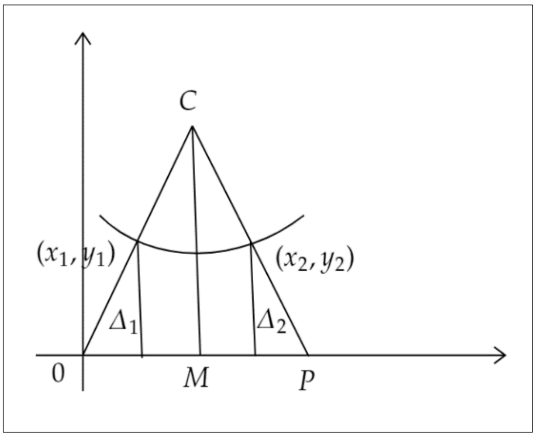

Fig 1: A part of the required region

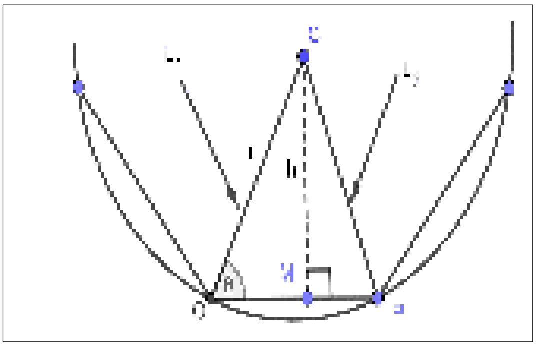

Fig 2: Computation of Intersection

$\angle OCP = \frac{2\pi}{n}$, so that $\theta = \pi - \frac{\pi}{n} - \frac{\pi}{2} = \left(\frac{n-2}{2n}\right)\pi$

By symmetry it is sufficient to study the part of the region that lies inside triangle OCP. Now from figure 1, equation of line L_1, given by.

$$y = mx. \qquad (2)$$

where $m = \tan\left[\left(\frac{n-2}{2n}\right)\pi\right]$. Similarly equation of line L_2, can be seen to be

$$L_2: y = -m(x-1) = m - mx, \qquad (3)$$

For the center $C = (x_0, y_0)$, we have $x_0 = \frac{1}{2}$. If r is the radius of the circumscribing circle then from the same triangle

$r = \frac{1}{2} cosec \frac{\pi}{n}$, $h = CM = \frac{1}{2}\cot\frac{\pi}{n}$, so that the center $C = \left(\frac{1}{2}, \frac{1}{2}\cot\frac{\pi}{n}\right)$.

To study the part of the region D, that lies inside the triangle OCP (see Figure 2), we need only compare the distance of a point (x,y) with the perpendicular distance from the side OP. A point (x,y) will fall in D, if the following condition holds.

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\cot\frac{\pi}{n}\right)^2 < y^2$$

or, $x^2 - x - \cot\frac{\pi}{n} y + \frac{1}{4} + \frac{1}{4}\cot^2\frac{\pi}{n} < 0$

or, $\left(x - \frac{1}{2}\right)^2 < \cot\frac{\pi}{n}\left(y - \frac{1}{4}\cot\frac{\pi}{n}\right)$,

Now, the equation

$$\left(x - \frac{1}{2}\right)^2 = \cot\frac{\pi}{n}\left(y - \frac{1}{4}\cot\frac{\pi}{n}\right), \qquad (4)$$

Describe the parabola with vertex at $\frac{1}{2}, \frac{1}{4}\cot\frac{\pi}{n}$). The parabola (4) intersect the line (2) where $y = mx$. Denoting $\cot\frac{\pi}{n}$ by a, the point of intersection(s) is/are are given by

$$\left(x - \frac{1}{2}\right)^2 = a\left(mx - \frac{a}{4}\right),$$

or, $4x^2 - 4(1+am)x + (1+a^2) = 0$. (5)

We note that both $a = \cot\frac{\pi}{n}$, and $m = \tan\left[\left(\frac{n-2}{2n}\right)\pi\right]$ approach infinity as n approach infinity. The roots of (5) are as follow.

$$x = \frac{(1+am) \pm \sqrt{(1+am)^2 - (1+a^2)}}{2}.$$

Denoting by x_1 the x-coordinates of the point of intersection of (2) and (4), from Figure 2, we have

$$x_1 = \frac{(1+am) - \sqrt{a^2(m^2-1) + 2am}}{2}.$$

In a similar way x_2, the point of intersection of (3) and (4), is given by.

$\left(x - \frac{1}{2}\right)^2 = a\left(y - \frac{a}{4}\right) = a\left(m - mx - \frac{a}{4}\right)$, or $4x^2 - 4(1 - am)x + (1 + a^2 - 4am) = 0$, yielding,

$$x_2 = \frac{(1-am) + \sqrt{a^2(1+m^2) + 2am}}{2}.$$

As both exponent x_1 and x_2 depend on parameters on n and a we shall henceforth denote them by $f_1(a,m)$ and $f_2(a,m)$. It is obvious $f_2(a,m) > f_1(a,m)$. Again from Figure 2:
Area of

$$\triangle_1 = \frac{1}{2} \times x_1 \times mx_1, = \frac{m}{2}\left[f_1(a,m)^2\right].$$

By symmetry
Area of $\triangle_1$ = Area of $\triangle_2$,
so that
$\triangle$ = Area ($\triangle_1$) + Area ($\triangle_2$),
$= m[f_1(a,m)]^{2,}$ (6)

The area of the curve (3) between the ordinates x_1 and x_2 is given by the integral

$$I = \frac{1}{a}\int_{x_1}^{x_2}\left[\left(x-\frac{1}{2}\right)^2+\frac{a^2}{4}\right]dx,$$

which is equal to

$$I = \frac{2}{a}\int_{x_1}^{\frac{1}{2}}\left[\left(x-\frac{1}{2}\right)^2+\frac{a^2}{4}\right]dx.$$

Evaluating the above integral, we obtain

$$I = \frac{1}{48}\left[-a^3m(3+m^2)+3a^2(1+m^2).S-3am.S^2+S^{3/2}\right]$$

$$S = \sqrt{a^2(m^2-1)+2am}.$$,

Where

If we confine our self to triangle OCP the area of the region D, that falls with in the polygon is given by

$$\frac{1}{4}\cot\frac{\pi}{n} - I - \triangle.$$

As the area of the polygon is $(\frac{1}{4}\cot\frac{\pi}{n})\times n$, the probability that a randomly chosen point inside the polygon falls within D, is given by

$$P(n) = \frac{\left(\frac{1}{4}\cot\frac{\pi}{n} - I - \triangle\right)n}{\frac{1}{4}n\cot\frac{\pi}{n}},$$

$$= 1-\frac{1}{4}\left(\frac{I+\triangle}{\cot\frac{\pi}{n}}\right),$$

$$= 1-\frac{\tan(\pi/n)}{4}(I+\triangle).$$

Using any Computer Algebra System like Mathematica, it can easily be verified that $P(n)$ approaches $\frac{1}{4}$ as n approaches infinity. This agrees with our intuition, since as n approaches infinity the polygon approaches a circle in the limit.

As an illustration of the geometry of the desired region, Figure 3 depicts how the region will look like for a square.

Remark: It will be interesting to explore the above problem with respect to other non-euclidean distance metrics such as the following:

Manhattan distance: $d_{\text{Manhattan}}((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$.

Chebyshev distance: $d_{\text{Chebyshev}}((x_1,y_1),(x_2,y_2)) = \max(|x_1 - x_2|,|y_1 - y_2|)$.

Minkowski distance: $d_{\text{Minkowski}}((x_1,y_1),(x_2,y_2)) = (|x_1 - x_2|^p + |y_1 - y_2|^p)^{1/p}$.

2

Hamming distance: $d_{\text{Hamming}}((x_1,y_1),(x_2,y_2)) = \sum_{i=1}^{2} [x_i != y_i]$

Code For Numerical Simulation

```
%Python code highlighting
\begin{lstlisting}[language=Python, caption= Python code]
import numpy as np
from random import uniform
from sympy import *
import scipy
from scipy.integrate import quad
import matplotlib.pyplot as plt
problist = []
for n in range(4,40):

def fm(n):
return tan(((n-2)/(2*n))*pi)

def fa(n):
return cot(pi/n)
def flimit(m,a):
m=fm(n)
a=fa(n)
return (1+a*m-sqrt(a**2*(m**2-1)+2*a*m))/2

def f(x):
a=fa(n)
return (x-1/2)**2+(a**2)/4

def intvalue(m,a):
m=fm(n)
a=fa(n)
s=sqrt(a**2*(m**2-1)+2*a*m)
return     1/24*(-a**3*m*(3+m**2)+3*a**2*(1+m**2)*s-3*a*m*s**2+s**3)
a=fa(n)
m=fm(n)
l=flimit(m,a)
u=0.5
#I = quad(f,l,u)
intt=intvalue(m,a)
#area=N(2/a*I[0])
area=N(2/a*intt)
trngle=N(m*l**2)
prob=1-tan(pi/n)*(area+trngle)*4
numprob=N(prob)
problist.append(numprob)
plt.plot(problist)
plt.xlabel("No. of sides of the polygon ")
plt.ylabel("Relative area where points are nearer to the center")
plt.show()
```

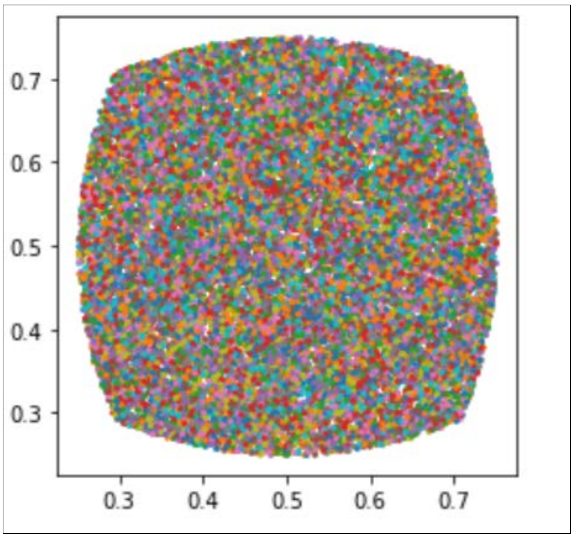

Fig 3: The region for n=4

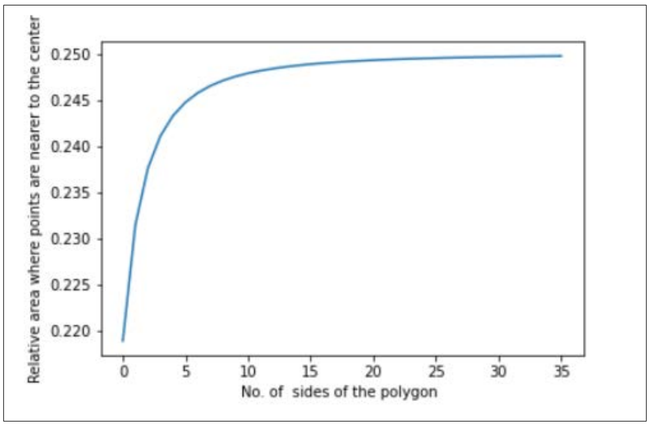

Fig 4: The variation of Probability

Conclusion

The paper presents a significant contribution to geometric probability by examining the likelihood of a random point falling within a specific region inside a regular polygon. By focusing on points closer to the polygon's center than its sides, the study not only adds to the theoretical understanding of geometric probability but also hints at practical applications in diverse fields such as image processing and spatial statistics. The findings, built on the foundational work in integral geometry, enrich our comprehension of spatial relationships in geometric figures, demonstrating the enduring relevance and adaptability of classical mathematical principles in modern contexts.

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