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Analysis of differential equations for the social sciences

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Abstract
This paper is designed to introduce social scientists to the fundamental concepts and uses of stochastic differential equations, as applied to several models of current interest: linear feedback, epidemics, and the cusp catastrophe. The approach taken here is to focus on the stationary probability density function of the model, which avoids most of the advanced mathematical techniques that are ordinarily required to describe the solutions of the equations.

Keywords: social sciences, fundamental, mathematical techniques

Introduction

The quantitative variables used in the social sciences tend to vary over time in ways that are apparently part systematic and part random. This is true in such diverse areas as economic indicators, vital statistics, opinion polls, and individual attitudes, behaviors, and characteristics. This significant random component plays havoc with the most fundamental tool of differential calculus: the derivative. If x(t) stands for a quantitative variable at time t, we should like to explain its derivative with respect to time, dx/dt, in terms of some function of x and other variables. This is impossible if the trajectory of x(t) is so irregular that it is nowhere differentiable, and yet this is exactly the situation found in the social sciences! Several strategies have been used whenever this problem has been confronted. They are as follows:

1. Divide time into discrete steps and use difference equations.
2. Abandon the ordinary mathematical definition of the derivative and use a statistical definition.

An example of strategy 2 is the Langevin (1908) Equation for certain kinds of Brownian motion. The Langevin Equation is worthy of attention because it was the first attempt to construct a stochastic differential equation, and because it worked in spite of its relatively ad hoc nature:

$$dx/dt = -\beta x + f(t) \qquad (1)$$

In this equation Langevin was trying to say that the acceleration (dx/dt) of a small particle suspended in a fluid is composed of a random force f(t) and a systematic (i.e. non-random) frictional force which is a linear function of the velocity of the particle (x).

At the heart of Itô's definition of a stochastic differential equation for a random variable x(t) is the specification of two functions. Intuitively described, the first, $\mu(x)$, specifies the expected rate of change in x(t) and the second, $\sigma^2(x)$, specifies the "variance" of this rate of change.

The word variance is in quotes here because it corresponds only loosely to the usual meaning of the term. The random input, wt, is assumed (in the Itô formulation) to be what is technically known as a Wiener Process, a mathematical idealization of Brownian notion. This assumption is quite reasonable for simple models of a very wide class of stochastic phenomena in the physical, biological and social sciences.

For those who would like precise definitions, the functions $\mu(x)$ and $\sigma^2(x)$ can be defined in terms of the conditional expectations of a stochastic process X(t) as:

$$\mu(x) = \lim_{h \to 0} \frac{E\{X(t+h) - X(t) \mid X(t) = x\}}{h} \tag{2}$$

$$\sigma^2(x) = \lim_{h \to 0} \frac{E\{\frac{1}{2}[X(t+h) - X(t)]^2 \mid X(t) = x\}}{h} \tag{3}$$

In the field of population genetics these are referred to as the "drift" and "diffusion" functions and this terminology will be used here. Having described all of its parts, we can now exhibit the general form of an Itô stochastic differential equation:

$$dx_t = \mu(x_t)dt + \sigma(x_t)dw_t. \tag{4}$$

Comparing (4) with (1), we can see that the Langevin Equation is specified by $\mu(x) = -\beta x$, $\sigma^2(x) = 1$, and $dw_t = f(t)dt$. Thus in the Itô form, Langevin's equation would read:

$$dx_t = -\beta x_t dt + dw_t. \tag{5}$$

The change in appearance between (1) and (5) is slight and superficial, but there is a major difference in the mathematical machinery that can be brought to bear on (5). It is not the appearance of an equation that is important, but what can be done with it.

Social scientists whose grounding is stronger in statistics than in calculus may gain some insight into the meaning of (4) by comparing it with its approximation in discrete time:

$$\Delta x_t = \mu(x_t)\Delta t + \sigma(x_t)u_t \tag{6}$$

Where u_t is a sequence of random variables, assumed to be normally distributed with zero mean and variance Δt, serially independent, and independent of x_t. The difference operator Δ has the usual meaning: $\Delta x_t = x_{t+\Delta t} - x_t$. Thus the discrete-time Langevin Equation would read $\Delta x_t = -\beta x_t \Delta t + u_t$, which is a first-order linear stochastic difference equation. The time series produced by this equation would be called "first-order autoregressive" by a statistician. Dynamic models based on equations of this type are now fairly common in economics, sociology, political science, and psychology.

It is not too difficult to show mathematically that if $\beta<0$ then the probability density function of x_t in the discrete-time Langevin Equation converges to the Normal density, no matter what its initial shape. Thus one possible explanation for the appearance of normally distributed empirical data is that individual cases are executing a random walk (Brownian motion) with a linear restoring force. Of course if $\mu(x)$ is anything other than linear or if $\sigma^2(x)$ is anything other than constant, then the probability density to which the population converges will not be Normal. This implies that by examining the shape of an empirical frequency distribution we can deduce a substantial amount of information about the nature of the functions $\mu(x)$ and $\sigma^2(x)$, assuming that the population is in statistical equilibrium. Let us see how this is done.

The evolution of the probability density function for a variable which behaves according to a stochastic differential equation is described, necessarily, by a partial differential equation. This is because the probability density function $f(x,t)$ is a function of both x and t (time). Interestingly, the evolution of f over time is entirely determined by the functions $\mu(x)$ and $\sigma^2(x)$:

$$\frac{\partial f}{\partial f} = \frac{\partial(\mu(x) f(x,t))}{\partial x} + \frac{\partial^2(\sigma^2(x) f(x,t))}{\partial x^2} \tag{7}$$

In some cases this equation, known as the Kolmogorov forward equation, can be explicitly solved.

$$f(x) = \frac{\psi}{\sigma^2} \exp\left[\int_{-\infty}^{x} \frac{\mu(s)}{\sigma^2(s)} ds\right], \tag{8}$$

Where the constant ψ is chosen so as to make

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

In a sense, Wright's Formula provides the stationary distribution for any process whose dynamics are described by an Itô stochastic differential equation. The value $f(x)dx$ is the probability that individual trajectories will pass through the interval between x and x+dx, once the population ensemble has reached statistical equilibrium. It therefore yields exactly the kind of statistical information which is most valuable to the social sciences.

We will now work through a complete example of the application of Wright's Formula to a familiar problem: a linear feedback system responding to a randomly changing environment. Consider a system whose state is described by a single variable x_t, with a "goal" G towards which it moves, given no exogenous disturbance. Because it is linear, its expected rate of change is proportional to its deviation from G: $\mu(x_t) = r(G-x_t)$, where $r > 0$ is the proportionality constant. Suppose further that the random disturbance to this system has a constant diffusion function.

Thus in Itô's formulation $\sigma^2(x) = \varepsilon$, where ε is a constant. Putting all this together, we get the Itô SDE:

$$dx_t = r(G - x_t)dt + \sqrt{\varepsilon} dw_t \tag{9}$$

What is the probability density function of x when statistical equilibrium is reached? To find out, we apply Wright's Formula, as follows:

$$f(x) = \frac{\psi}{\varepsilon} \exp\left[\int_{-\infty}^{x} \frac{r(G-s)}{\varepsilon} ds\right]$$

$$\Rightarrow f(x) = \frac{\psi}{\varepsilon} \exp\left[-\frac{(x-G)^2}{2\varepsilon / r}\right]. \tag{10}$$

Conclusion

Stochastic differential equations appear to hold considerable promise for the social sciences. First, they provide a powerful way to express the stochastic and deterministic components of a model on an equal basis. This ability overcomes the major drawback of ordinary differential equations for use in the social sciences.

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