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Integer solutions on the hyperbola $y^2 = 32x^2 + 36$

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Abstract
The binary quadratic equation represented by the positive pellian $y^2 = 32x^2 + 36$ is analysed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas.

Keywords: Binary quadratic, hyperbola, parabola, pell equation, integral solutions

Introduction
The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integer solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 32x^2 + 36$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

Method of analysis
The diophantine equation under consideration is

$$y^2 = 32x^2 + 36 \qquad (1)$$

The smallest positive integer solution (x_0, y_0) of (1) is

$$x_0 = 3\ ,\ y_0 = 18$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 32x^2 + 1 \qquad (2)$$

whose smallest positive integer solution is

$$\tilde{x}_0 = 3\ ,\ \tilde{y}_0 = 17$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by

$$\tilde{y}_n + \sqrt{32}\tilde{x}_n = \left(17 + 3\sqrt{32}\right)^{n+1}, \text{ where } n = 0,1,2,... \qquad (3)$$

Since irrational roots occur in pairs, we have

$$\tilde{y}_n - \sqrt{32}\tilde{x}_n = \left(17 - 3\sqrt{32}\right)^{n+1}, \text{ where } n = 0,1,2,... \qquad (4)$$

From (3) and (4), solving for $(\tilde{x}_n, \tilde{y}_n)$, we have

$$\tilde{y}_n = \frac{1}{2} f_n\ ,\ \tilde{x}_n = \frac{1}{2\sqrt{32}} g_n$$

where

$$f_n = \left(17 + 3\sqrt{32}\right)^{n+1} + \left(17 - 3\sqrt{32}\right)^{n+1}\ ,\ g_n = \left(17 + 3\sqrt{32}\right)^{n+1} - \left(17 - 3\sqrt{32}\right)^{n+1}$$

Applying Brahmagupta Lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions to (1) are given by

$$2\sqrt{32}x_{n+1} = 3\sqrt{32}f_n + 18g_n \quad 2y_{n+1} = 18f_n + 3\sqrt{32}g_n$$

A few numerical examples are given in the Table: 1 below

Table 1: Numerical Examples

n	x_{n+1}	y_{n+1}
-1	3	18
0	105	594
1	3567	20178
2	121173	685458

A few interesting relations among the solutions are given below

- $68y_{n+2} - 2y_{n+3} - 2y_{n+1} = 0$
- $x_{n+3} - 34x_{n+2} + x_{n+1} = 0$
- $3y_{n+1} - x_{n+2} + 17x_{n+1} = 0$
- $3y_{n+2} - 17x_{n+2} + x_{n+1} = 0$
- $3y_{n+3} - 577x_{n+2} + 17x_{n+1} = 0$
- $102y_{n+1} - x_{n+3} + 577x_{n+1} = 0$
- $6y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $17y_{n+1} - y_{n+2} + 96x_{n+1} = 0$
- $17y_{n+3} - 577y_{n+2} - 96x_{n+1} = 0$
- $577x_{n+3} - 102y_{n+3} - x_{n+1} = 0$
- $577y_{n+1} - y_{n+3} + 3264x_{n+1} = 0$
- $577y_{n+2} - 289y_{n+3} + 1829568x_{n+1} = 0$
- $3y_{n+1} - 17x_{n+3} + 577x_{n+2} = 0$
- $3y_{n+2} - x_{n+3} + 17x_{n+2} = 0$
- $3y_{n+3} - 17x_{n+3} + x_{n+2} = 0$
- $17y_{n+2} - y_{n+1} - 96x_{n+2} = 0$
- $y_{n+3} - y_{n+1} - 192x_{n+2} = 0$
- $y_{n+3} - 17y_{n+2} - 96x_{n+2} = 0$
- $577y_{n+2} - 17y_{n+1} - 96x_{n+3} = 0$
- $17y_{n+3} - y_{n+2} - 96x_{n+3} = 0$
- $y_{n+1} - 577y_{n+3} + 3264x_{n+3} = 0$
- $y_{n+3} - 34y_{n+2} + y_{n+1} = 0$

Each of the following expressions represents a cubical integer:

- $\frac{1}{108}((36x_{3n+4} - 1188x_{3n+3}) + 3(36x_{n+2} - 1188x_{n+1}))$
- $\frac{1}{306}((3x_{3n+5} - 3363x_{3n+3}) + 3(3x_{n+3} - 3363x_{n+1}))$
- $\frac{1}{102}((6y_{3n+4} - 1120x_{3n+3}) + 3(6y_{n+2} - 1120x_{n+1}))$
- $\frac{1}{1731}((3y_{3n+5} - 19024x_{3n+3}) + 3(3y_{n+3} - 19024x_{n+1}))$

- $\frac{1}{9}\left(\left(99x_{3n+5}-3363x_{3n+4}\right)+3\left(99x_{n+3}-3363x_{n+2}\right)\right)$
- $\frac{1}{51}\left(\left(99y_{3n+3}-16x_{3n+4}\right)+3\left(99y_{n+1}-16x_{n+2}\right)\right)$
- $\frac{1}{3}\left(\left(99y_{3n+4}-560x_{3n+4}\right)+3\left(99y_{n+2}-560x_{n+2}\right)\right)$
- $\frac{1}{51}\left(\left(99y_{3n+5}-19024x_{3n+4}\right)+3\left(99y_{n+3}-19024x_{n+2}\right)\right)$
- $\frac{1}{1731}\left(\left(3363y_{3n+3}-16x_{3n+5}\right)+3\left(3363y_{n+1}-16x_{n+3}\right)\right)$
- $\frac{1}{51}\left(\left(3363y_{3n+4}-560x_{3n+5}\right)+3\left(3363y_{n+2}-560x_{n+3}\right)\right)$
- $\frac{1}{3}\left(\left(3363y_{3n+5}-19024x_{3n+5}\right)+3\left(3363y_{n+3}-19024x_{n+3}\right)\right)$
- $\frac{1}{18}\left(\left(35y_{3n+3}-y_{3n+4}\right)+3\left(35y_{n+1}-y_{n+2}\right)\right)$
- $\frac{1}{612}\left(\left(1189y_{3n+3}-y_{3n+5}\right)+3\left(1189y_{n+1}-y_{n+3}\right)\right)$
- $\frac{1}{18}\left(\left(1189y_{3n+4}-35y_{3n+5}\right)+3\left(1189y_{n+2}-35y_{n+3}\right)\right)$

Each of the following expressions represents a bi-quadratic integer:

- $\frac{1}{108}\left[\left(36x_{4n+5}-1188x_{4n+4}\right)+4\left(36x_{2n+3}-1188x_{2n+2}+216\right)-216\right]$
- $\frac{1}{306}\left(\left(3x_{4n+6}-3363x_{4n+4}\right)+4\left(3x_{2n+4}-3363x_{2n+2}+612\right)-612\right)$
- $\frac{1}{102}\left(\left(6y_{4n+5}-1120x_{4n+4}\right)+4\left(6y_{2n+3}-1120x_{2n+2}+204\right)-204\right)$
- $\frac{1}{1731}\left(\left(3y_{4n+6}-19024x_{4n+4}\right)+4\left(3y_{2n+4}-19024x_{2n+2}+3462\right)-3462\right)$
- $\frac{1}{9}\left(\left(99x_{4n+6}-3363x_{4n+5}\right)+4\left(99x_{2n+4}-3363x_{2n+3}+18\right)-18\right)$
- $\frac{1}{51}\left(\left(99y_{4n+4}-16x_{4n+5}\right)+4\left(99y_{2n+2}-16x_{2n+3}+102\right)-102\right)$
- $\frac{1}{3}\left(\left(99y_{4n+5}-560x_{4n+5}\right)+4\left(99y_{2n+3}-560x_{2n+3}+6\right)-6\right)$
- $\frac{1}{51}\left(\left(99y_{4n+6}-19024x_{4n+5}\right)+4\left(99y_{2n+4}-19024x_{2n+3}+102\right)-102\right)$
- $\frac{1}{1731}\left(\left(3363y_{4n+4}-16x_{4n+6}\right)+4\left(3363y_{2n+2}-16x_{2n+4}+3462\right)-3462\right)$
- $\frac{1}{51}\left(\left(3363y_{4n+5}-560x_{4n+6}\right)+4\left(3363y_{2n+3}-560x_{2n+4}+102\right)-102\right)$
- $\frac{1}{3}\left(\left(3363y_{4n+6}-19024x_{4n+6}\right)+4\left(3363y_{2n+4}-19024y_{2n+4}+6\right)-6\right)$
- $\frac{1}{18}\left(\left(35y_{4n+4}-y_{4n+5}\right)+4\left(35y_{2n+2}-y_{2n+3}+36\right)-36\right)$
- $\frac{1}{612}\left(\left(1189y_{4n+4}-y_{4n+6}\right)+4\left(1189y_{2n+2}-y_{2n+4}+1224\right)-1224\right)$

- $\frac{1}{18}\left(\left(1189y_{4n+5}-35y_{4n+6}\right)+4\left(1189y_{2n+3}-35y_{2n+4}+36\right)-36\right)$

Each of the following expressions represents a Nasty Number

- $\frac{1}{18}\left(36x_{2n+3}-1188x_{2n+2}+216\right)$
- $\frac{1}{51}\left(3x_{2n+4}-3363x_{2n+2}+612\right)$
- $\frac{1}{17}\left(6y_{2n+3}-1120x_{2n+2}+204\right)$
- $\frac{2}{577}\left(3y_{2n+4}-19024x_{2n+2}+3462\right)$
- $\frac{2}{3}\left(99x_{2n+4}-3363x_{2n+3}+18\right)$
- $\frac{2}{17}\left(99y_{2n+2}-16x_{2n+3}+102\right)$
- $2\left(99y_{2n+3}-560x_{2n+3}+6\right)$
- $\frac{2}{17}\left(99y_{2n+4}-19024x_{2n+3}+102\right)$
- $\frac{2}{577}\left(3363y_{2n+2}-16x_{2n+4}+3462\right)$
- $\frac{2}{17}\left(3363y_{2n+3}-560x_{2n+4}+102\right)$
- $2\left(3363y_{2n+4}-19024x_{2n+4}+6\right)$
- $\frac{1}{3}\left(35y_{2n+2}-y_{2n+3}+36\right)$
- $\frac{1}{102}\left(1189y_{2n+2}-y_{2n+4}+1224\right)$
- $\frac{1}{3}\left(1189y_{2n+3}-35y_{2n+4}+36\right)$

Remarkable observations

(i) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in Table: 2 below:

Table 2: Hyperbolas

S. No	Hyperbolas	$\left(X_n, Y_n\right)$
1	$32Y_n^2-X_n^2=1492992$	$\left(\left(6720x_{n+1}-192x_{n+2}\right),\left(36x_{n+2}-1188x_{n+1}\right)\right)$
2	$32Y_n^2-X_n^2=11985408$	$\left(\left(19024x_{n+1}-16x_{n+3}\right),\left(3x_{n+3}-3363x_{n+1}\right)\right)$
3	$32Y_n^2-X_n^2=1331712$	$\left(\left(6336x_{n+1}-32y_{n+2}\right),\left(6y_{n+2}-1120x_{n+1}\right)\right)$
4	$32Y_n^2-X_n^2=383534208$	$\left(\left(107616x_{n+1}-16y_{n+3}\right),\left(3y_{n+3}-19024x_{n+1}\right)\right)$
5	$32Y_n^2-X_n^2=10368$	$\left(\left(19024x_{n+2}-560x_{n+3}\right),\left(99x_{n+3}-3363x_{n+2}\right)\right)$
6	$32Y_n^2-X_n^2=332928$	$\left(\left(96x_{n+2}-560y_{n+1}\right),\left(99y_{n+1}-16x_{n+2}\right)\right)$
7	$32Y_n^2-X_n^2=1152$	$\left(\left(3168x_{n+2}-560y_{n+2}\right),\left(99y_{n+2}-560x_{n+2}\right)\right)$
8	$32Y_n^2-X_n^2=332928$	$\left(\left(107616x_{n+2}-560y_{n+3}\right),\left(99y_{n+3}-19024x_{n+2}\right)\right)$
9	$32Y_n^2-X_n^2=383534208$	$\left(\left(96x_{n+3}-19024y_{n+1}\right),\left(3363y_{n+1}-16x_{n+3}\right)\right)$

10	$32Y_n^{\ 2} - X_n^{\ 2} = 332928$	$((3168x_{n+3} - 19024y_{n+2}),(3363y_{n+2} - 560x_{n+3}))$
11	$32Y_n^{\ 2} - X_n^{\ 2} = 1152$	$((107616x_{n+3} - 19024y_{n+3}),(3363y_{n+3} - 19024x_{n+3}))$
12	$32Y_n^{\ 2} - X_n^{\ 2} = 41472$	$((6y_{n+2} - 198y_{n+1}),(35y_{n+1} - y_{n+2}))$
13	$32Y_n^{\ 2} - X_n^{\ 2} = 47941632$	$((6y_{n+3} - 6726y_{n+1}),(1189y_{n+1} - y_{n+3}))$
14	$32Y_n^{\ 2} - X_n^{\ 2} = 41472$	$(198y_{n+3} - 6726y_{n+2}),(1189y_{n+2} - 35y_{n+3})$

(ii) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table: 3 below:

Table 3: Parabolas

S. No	Parabolas	(X_n, Y_n)
1	$3456Y_n - X_n^{\ 2} = 1492992$	$[(6720x_{n+1} - 192x_{n+2}),(36x_{2n+3} - 1188x_{2n+2} + 216)]$
2	$9792Y_n - X_n^{\ 2} = 11985408$	$[(19024x_{n+1} - 16x_{n+3}),(3x_{2n+4} - 3363x_{2n+2} + 612)]$
3	$3264Y_n - X_n^{\ 2} = 1331712$	$[(6336x_{n+1} - 32y_{n+2}),(6y_{2n+3} - 1120x_{2n+2} + 204)]$
4	$55392Y_n - X_n^{\ 2} = 383534208$	$[(107616x_{n+1} - 16y_{n+3}),(3y_{2n+4} - 19024x_{2n+2} + 3462)]$
5	$288Y_n - X_n^{\ 2} = 10368$	$[(19024x_{n+2} - 560x_{n+3}),(99x_{2n+4} - 3363x_{2n+3} + 18)]$
6	$1632Y_n - X_n^{\ 2} = 332928$	$[(96x_{n+2} - 560y_{n+1}),(99y_{2n+2} - 16x_{2n+3} + 102)]$
7	$96Y_n - X_n^{\ 2} = 1152$	$[(3168x_{n+2} - 560y_{n+2}),(99y_{2n+3} - 560y_{2n+3} + 6)]$
8	$1632Y_n - X_n^{\ 2} = 332928$	$[(107616x_{n+2} - 560y_{n+3}),(99y_{2n+4} - 19024x_{2n+3} + 102)]$
9	$55392Y_n - X_n^{\ 2} = 383534208$	$[(96x_{n+3} - 19024y_{n+1}),(3363y_{2n+2} - 16x_{2n+4} + 3462)]$
10	$1632Y_n - X_n^{\ 2} = 332928$	$[(3168x_{n+3} - 19024y_{n+2}),(3363y_{2n+3} - 560x_{2n+4} + 102)]$
11	$96Y_n - X_n^{\ 2} = 1152$	$[(107616x_{n+3} - 19024y_{n+3}),(3363y_{2n+4} - 19024x_{2n+4} + 6)]$
12	$576Y_n - X_n^{\ 2} = 41472$	$[(6y_{n+2} - 198y_{n+1}),(35y_{2n+2} - y_{2n+3} + 36)]$
13	$19584Y_n - X_n^{\ 2} = 47941632$	$[(6y_{n+3} - 6726y_{n+1}),(1189y_{2n+2} - y_{2n+4} + 1224)]$
14	$576Y_n - X_n^{\ 2} = 41472$	$[(198y_{n+3} - 6726y_{n+2}),(1189y_{2n+3} - 35y_{2n+4} + 36)]$

Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2 = 32x^2 + 36$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of pell equations and determine their integer solutions along with suitable properties.

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