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Generation formula for solutions to special ternary quadratic diophantine equations representing cones

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Abstract
Knowing a solution of the considered ternary quadratic diophantine equation, a general formula for generating sequence of solutions based on the given solution is illustrated.

Keywords: ternary quadratic, generation of solutions

Introduction

The subject of diophantine equations in number theory has attracted many mathematicians since antiquity. It is well-known that a diophantine equation is a polynomial equation in two or more unknowns with integer coefficients for which integer solutions are required. An integer solution is a solution such that all the unknowns in the equation take integer values. An extension of ordinary integers into complex numbers is the gaussian integers. A gaussian integer is a complex number whose real and imaginary parts are both integers. It is quite obvious that diophantine equations are rich in variety and there are methods available to obtain solutions either in real integers or in gaussian integers.

A natural question that arises now is, whether a general formula for generating sequence of solutions based on the given solution can be obtained? In this context, one may refer [1-7]. The main thrust of this communication is to show that the answer to the above question is in the affirmative in the case of the following ternary quadratic diophantine equations, each representing a cone.

Cone 1

The ternary quadratic diophantine equation under consideration is

$$x^2 + y^2 = 5z^2 \tag{1}$$

Let (x_0, y_0, z_0) be any solution of (1).

The solution may be in real integers or in gausssian integers or in irrational numbers.

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = 2h - 3x_0\ ,\ y_1 = 2h - 3y_0\ ,\ z_1 = 3z_0 + h \tag{2}$$

in which h is an unknown to be determined.

Substitution of (2) in (1) gives

$$h = 4x_0 + 4y_0 + 10\ z_0 \tag{3}$$

Using (3) in (2), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\ (x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 5 & 8 & 20 \\ 8 & 5 & 20 \\ 4 & 4 & 13 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{Y_n + (-3)^{n+1}}{2} & \frac{Y_n - (-3)^{n+1}}{2} & 5X_n \\ \frac{Y_n - (-3)^{n+1}}{2} & \frac{Y_n + (-3)^{n+1}}{2} & 5X_n \\ X_n & X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, n = 0,1,2,....$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = 10X^2 + 9$

That is,

$$X_n = X_0 Y_{n-1} + Y_0 X_{n-1}$$

$$Y_n = Y_0 Y_{n-1} + 10 X_0 X_{n-1}, \quad n = 0,1,2,...., \quad X_{-1} = 0, \; Y_{-1} = 1$$

Cone: 2

The ternary quadratic diophantine equation under consideration is

$$x^2 + y^2 = 6z^2 \tag{1}$$

Let (x_0, y_0, z_0) be any solution of (1).

The solution may be in real integers or in gausssian integers or in irrational numbers.

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = 2h - x_0, \; y_1 = 2h - y_0, \; z_1 = z_0 + h \tag{2}$$

In which h is an unknown to be determined.

Substitution of (2) in (1) gives

$$h = 2x_0 + 2y_0 + 6z_0 \tag{3}$$

Using (3) in (2), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M (x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 3 & 4 & 12 \\ 4 & 3 & 12 \\ 2 & 2 & 7 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{Y_n - (-1)^{n}}{2} & \frac{Y_n + (-1)^{n}}{2} & 6X_n \\ \frac{Y_n + (-1)^{n}}{2} & \frac{Y_n - (-1)^{n}}{2} & 6X_n \\ X_n & X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, n = 0,1,2,....$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = 12X^2 + 1$

That is,

$$Y_n = \frac{1}{2}\left(\left(7 + 2\sqrt{12}\right)^{n+1} + \left(7 - 2\sqrt{12}\right)^{n+1}\right)$$

$$X_n = \frac{1}{2\sqrt{12}}\left(\left(7 + 2\sqrt{12}\right)^{n+1} - \left(7 - 2\sqrt{12}\right)^{n+1}\right)$$

Cone: 3

The ternary quadratic diophantine equation under consideration is

$$x^2 + y^2 = 10z^2 \quad (1)$$

Let (x_0, y_0, z_0) be any solution of (1).

The solution may be in real integers or in gausssian integers or in irrational numbers.

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = x_0 + 2h,\ y_1 = y_0 + 2h,\ z_1 = h - z_0 \quad (2)$$

in which h is an unknown to be determined.

Substitution of (2) in (1) gives

$$h = 2x_0 + 2y_0 + 10z_0 \quad (3)$$

Using (3) in (2), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\,(x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 5 & 4 & 20 \\ 4 & 5 & 20 \\ 2 & 2 & 9 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{Y_n + 1}{2} & \frac{Y_n - 1}{2} & 10X_n \\ \frac{Y_n - 1}{2} & \frac{Y_n + 1}{2} & 10X_n \\ X_n & X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix},\ n = 0, 1, 2, \ldots.$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = 20X^2 + 1$

That is,

$$Y_n = \frac{1}{2}\left(\left(9 + 2\sqrt{20}\right)^{n+1} + \left(9 - 2\sqrt{20}\right)^{n+1}\right)$$

$$X_n = \frac{1}{2\sqrt{20}}\left(\left(9 + 2\sqrt{20}\right)^{n+1} - \left(9 - 2\sqrt{20}\right)^{n+1}\right)$$

Cone: 4

The ternary quadratic diophantine equation under consideration is

$$x^2 + y^2 = 22z^2 \quad (1)$$

Let (x_0, y_0, z_0) be any solution of (1).
The solution may be in real integers or in gausssian integers or in irrational numbers.
Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = x_0 + 4h \ , \ y_1 = y_0 + 2h \ , \ z_1 = h - z_0 \tag{2}$$

in which h is an unknown to be determined.
Substitution of (2) in (1) gives

$$h = 4x_0 + 2y_0 + 22z_0 \tag{3}$$

Using (3) in (2), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\,(x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 17 & 8 & 88 \\ 8 & 5 & 44 \\ 4 & 2 & 21 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{4Y_n + 1}{5} & \frac{2(Y_n - 1)}{5} & 44X_n \\ \frac{2(Y_n - 1)}{5} & \frac{Y_n + 4}{5} & 22X_n \\ 2X_n & X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \quad , n = 0, 1, 2, ...$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = 110X^2 + 1$
That is,

$$Y_n = \frac{1}{2}\left(\left(21 + 2\sqrt{110}\right)^{n+1} + \left(21 - 2\sqrt{110}\right)^{n+1} \right)$$

$$X_n = \frac{1}{2\sqrt{110}}\left(\left(21 + 2\sqrt{110}\right)^{n+1} - \left(21 - 2\sqrt{110}\right)^{n+1} \right)$$

Cone: 5
The ternary quadratic diophantine equation under consideration is

$$5x^2 + y^2 = z^2 \tag{1}$$

Let (x_0, y_0, z_0) be any solution of (1).
The solution may be in real integers or in gausssian integers or in irrational numbers.

Choice: 1
Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = h - x_0 \ , \ y_1 = h - y_0 \ , \ z_1 = z_0 + 2h \tag{2}$$

in which h is an unknown to be determined.
Substitution of (2) in (1) gives

$$h = 5x_0 + y_0 + 2z_0 \tag{3}$$

Using (3) in (2), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\,(x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 4 & 1 & 2 \\ 5 & 0 & 2 \\ 10 & 2 & 5 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{5Y_n - (-1)^n}{6} & \frac{Y_n + (-1)^n}{6} & X_n \\ \frac{5\left(Y_n + (-1)^n\right)}{6} & \frac{Y_n - 5(-1)^n}{6} & X_n \\ 5X_n & X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \quad n = 0, 1, 2, \ldots.$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = 6X^2 + 1$
That is,

$$Y_n = \frac{1}{2}\left(\left(5 + 2\sqrt{6}\right)^{n+1} + \left(5 - 2\sqrt{6}\right)^{n+1}\right)$$

$$X_n = \frac{1}{2\sqrt{6}}\left(\left(5 + 2\sqrt{6}\right)^{n+1} - \left(5 - 2\sqrt{6}\right)^{n+1}\right)$$

Choice: 2

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = 3x_0 + h\,,\ y_1 = 3y_0 + h\,,\ z_1 = 3h - 3z_0 \tag{4}$$

in which h is an unknown to be determined.
Substitution of (4) in (1) gives

$$h = 10\,x_0 + 2y_0 + 6z_0 \tag{5}$$

Using (5) in (4), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\,(x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 13 & 2 & 6 \\ 10 & 5 & 6 \\ 30 & 6 & 15 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{5Y_n + 3^{n+1}}{6} & \frac{Y_n - 3^{n+1}}{6} & X_n \\ \frac{5(Y_n - 3^{n+1})}{6} & \frac{Y_n + 5(3)^{n+1}}{6} & X_n \\ 5X_n & X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, n = 0, 1, 2, ...$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = 6X^2 + 9$

That is,

$$X_n = X_0 Y_{n-1} + Y_0 X_{n-1}$$

$$Y_n = Y_0 Y_{n-1} + 6X_0 X_{n-1}, \; n = 0, 1, 2, 3, ... \quad , X_{-1} = 0, \; Y_{-1} = 1$$

Cone: 6

The ternary quadratic diophantine equation under consideration is

$$6x^2 + y^2 = z^2 \quad (1)$$

Let (x_0, y_0, z_0) be any solution of (1).

The solution may be in real integers or in gausssian integers or in irrational numbers.

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = x_0 + h, \; y_1 = y_0 + h, \; z_1 = 3h - z_0 \quad (2)$$

in which h is an unknown to be determined.

Substitution of (2) in (1) gives

$$h = 6x_0 + y_0 + 3z_0 \quad (3)$$

Using (3) in (2), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\,(x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 7 & 1 & 3 \\ 6 & 2 & 3 \\ 18 & 3 & 8 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{6Y_n + 1}{7} & \frac{Y_n - 1}{7} & X_n \\ \frac{6(Y_n - 1)}{7} & \frac{Y_n + 6}{7} & X_n \\ 6X_n & X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, n = 0, 1, 2, ...$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = 7X^2 + 1$

That is,

$$Y_n = \frac{1}{2}\left(\left(8+3\sqrt{7}\right)^{n+1} + \left(8-3\sqrt{7}\right)^{n+1} \right)$$

$$X_n = \frac{1}{2\sqrt{7}}\left(\left(8+3\sqrt{7}\right)^{n+1} - \left(8-3\sqrt{7}\right)^{n+1} \right)$$

Cone: 7

The ternary quadratic diophantine equation under consideration is

$$7x^2 + y^2 = z^2 \tag{1}$$

Let (x_0, y_0, z_0) be any solution of (1).

The solution may be in real integers or in gausssian integers or in irrational numbers.

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = x_0 + h\ ,\ y_1 = y_0 + h\ ,\ z_1 = 3h - z_0 \tag{2}$$

In which h is an unknown to be determined

Substitution of (2) in (1) gives

$$h = 14x_0 + 2y_0 + 6z_0 \tag{3}$$

Using (3) in (2), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\,(x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 15 & 2 & 6 \\ 14 & 3 & 6 \\ 42 & 6 & 17 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{7Y_n+1}{8} & \frac{Y_n-1}{8} & X_n \\ \frac{7(Y_n-1)}{8} & \frac{Y_n+7}{8} & X_n \\ 7X_n & X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \quad , n = 0, 1, 2, ...$$

where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = 8X^2 + 1$

That is,

$$Y_n = \frac{1}{2}\left(\left(17+6\sqrt{8}\right)^{n+1} + \left(17-6\sqrt{8}\right)^{n+1} \right)$$

$$X_n = \frac{1}{2\sqrt{8}}\left(\left(17+6\sqrt{8}\right)^{n+1} - \left(17-6\sqrt{8}\right)^{n+1} \right)$$

Cone: 8

The ternary quadratic diophantine equation under consideration is

$$3x^2 + 4y^2 = z^2 \tag{1}$$

Let (x_0, y_0, z_0) be any solution of (1).
The solution may be in real integers or in gausssian integers or in irrational numbers.
Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = x_0 + h \ , \ y_1 = y_0 + h \ , \ z_1 = 3h - z_0 \tag{2}$$

in which h is an unknown to be determined.
Substitution of (2) in (1) gives

$$h = 3x_0 + 4y_0 + 3z_0 \tag{3}$$

Using (3) in (2), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\,(x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 4 & 4 & 3 \\ 3 & 5 & 3 \\ 9 & 12 & 8 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{3Y_n + 4}{7} & \frac{4(Y_n - 1)}{7} & X_n \\ \frac{3(Y_n - 1)}{7} & \frac{4Y_n + 3}{7} & X_n \\ 3X_n & 4X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \ , n = 0, 1, 2, \ldots.$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = 7X^2 + 1$
That is,

$$Y_n = \frac{1}{2}\left(\left(8 + 3\sqrt{7}\right)^{n+1} + \left(8 - 3\sqrt{7}\right)^{n+1} \right)$$

$$X_n = \frac{1}{2\sqrt{7}}\left(\left(8 + 3\sqrt{7}\right)^{n+1} - \left(8 - 3\sqrt{7}\right)^{n+1} \right)$$

Cone: 9
The ternary quadratic diophantine equation under consideration is

$$x^2 + y^2 = \left(k^2 + 2k + 4\right)z^2 \ , \ k > 0 \tag{1}$$

Let (x_0, y_0, z_0) be any solution of (1).
The solution may be in real integers or in gausssian integers or in irrational numbers.

Choice: 1
Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = x_0 + (k+1)h \;,\; y_1 = y_0 + h \;,\; z_1 = h - z_0 \tag{2}$$

in which h is an unknown to be determined.
Substitution of (2) in (1) gives

$$h = (k+1)x_0 + y_0 + \left(k^2 + 2k + 4\right)z_0 \tag{3}$$

Using (3) in (2), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\,(x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} k^2+2k+2 & k+1 & k^3+3k^2+6k+4 \\ k+1 & 2 & k^2+2k+4 \\ k+1 & 1 & k^2+2k+3 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \dfrac{(k+1)^2 Y_n + 1}{k^2+2k+2} & \dfrac{(k+1)(Y_n - 1)}{k^2+2k+2} & \left(k^3+3k^2+6k+4\right)X_n \\ \dfrac{(k+1)(Y_n - 1)}{k^2+2k+2} & \dfrac{Y_n + (k+1)^2}{k^2+2k+2} & \left(k^2+2k+4\right)X_n \\ (k+1)X_n & X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \;,\; n = 0, 1, 2, \ldots.$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = \left(k^4 + 4k^3 + 10k^2 + 12k + 8\right)X^2 + 1$
That is,

$$Y_n = \frac{1}{2}\left(\left(k^2+2k+3+\sqrt{k^4+4k^3+10k^2+12k+8}\right)^{n+1} + \left(k^2+2k+3-\sqrt{k^4+4k^3+10k^2+12k+8}\right)^{n+1} \right)$$

$$X_n = \frac{1}{2\sqrt{k^4+4k^3+10k^2+12k+8}}\left(\left(k^2+2k+3+\sqrt{k^4+4k^3+10k^2+12k+8}\right)^{n+1} - \left(k^2+2k+3-\sqrt{k^4+4k^3+10k^2+12k+8}\right)^{n+1} \right)$$

Choice: 2
Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = x_0 + h \;,\; y_1 = y_0 + (k+1)h \;,\; z_1 = h - z_0 \tag{4}$$

in which h is an unknown to be determined.
Substitution of (4) in (1) gives

$$h = x_0 + (k+1)y_0 + \left(k^2 + 2k + 4\right)z_0 \tag{5}$$

Using (5) in (4), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\,(x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 2 & k+1 & k^2+2k+4 \\ k+1 & k^2+2k+2 & k^3+3k^2+6k+4 \\ 1 & k+1 & k^2+2k+3 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{Y_n + (k+1)^2}{k^2+2k+2} & \frac{(k+1)(Y_n-1)}{k^2+2k+2} & (k^2+2k+4)X_n \\ \frac{(k+1)(Y_n-1)}{k^2+2k+2} & \frac{(k+1)^2 Y_n + 1}{k^2+2k+2} & (k^3+3k^2+6k+4)X_n \\ X_n & (k+1)X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, n = 0,1,2,....$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = (k^4 + 4k^3 + 10k^2 + 12k + 8)X^2 + 1$

Cone: 10

The ternary quadratic diophantine equation under consideration is

$$x^2 + (k+1)y^2 = (k+3)z^2 \;, \; k > 0 \qquad (1)$$

Let (x_0, y_0, z_0) be any solution of (1).

The solution may be in real integers or in gausssian integers or in irrational numbers.

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = x_0 + h \;, \; y_1 = y_0 + h \;, \; z_1 = h - z_0 \qquad (2)$$

in which h is an unknown to be determined.

Substitution of (2) in (1) gives

$$h = 2x_0 + (2k+2)y_0 + (2k+6)z_0 \qquad (3)$$

Using (3) in (2), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M\,(x_0, y_0, z_0)^t$$

Where t is the transpose and

$$M = \begin{pmatrix} 3 & 2k+2 & 2k+6 \\ 2 & 2k+3 & 2k+6 \\ 2 & 2k+2 & 2k+5 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{Y_n + k + 1}{k+2} & \frac{(k+1)(Y_n - 1)}{k+2} & (k+3)X_n \\ \frac{Y_n - 1}{k+2} & \frac{(k+1)Y_n + 1}{k+2} & (k+3)X_n \\ X_n & (k+1)X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \quad , n = 0, 1, 2, ...$$

Where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = (k^2 + 5k + 6)X^2 + 1$

That is,

$$Y_n = \frac{1}{2}\left(\left(2k + 5 + 2\sqrt{k^2 + 5k + 6}\right)^{n+1} + \left(2k + 5 - 2\sqrt{k^2 + 5k + 6}\right)^{n+1} \right)$$

$$X_n = \frac{1}{2\sqrt{k^2 + 5k + 6}}\left(\left(2k + 5 + 2\sqrt{k^2 + 5k + 6}\right)^{n+1} - \left(2k + 5 - 2\sqrt{k^2 + 5k + 6}\right)^{n+1} \right)$$

In conclusion, one may attempt to obtain generation formula for other choices of quadratic diophantine equations with multiple variables

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