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Simple and accurate approach to teaching linear algebra

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Abstract
In this Studies, we develop the philosophical and practical background for teaching an elementary Linear Algebra course from the requirements that are particular to the subject. It mixes the inner workings and logic of Linear Algebra and matrices with concepts, hands-on computational schemes, and applications for a satisfying comprehensive teaching and learning experience.

Keywords: linear algebra

Introduction

For a wholistic, unified, and balanced approach to teaching Linear Algebra, we should start with the key notion of Linear Algebra. In our opinion, Linear Algebra essentially deals with: vectors, geometry, linear transformations, and matrices.

And in essence, Linear Algebra is governed by an equation, namely the algebraic linearity condition

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

This equation contains the main ingredients of the subject. Namely

a) vectors x and y,
b) linear combinations $\alpha x + \beta y$, or geometry, for scalars α and β and vectors x, y,
c) and a linear transformation f between two vector spaces.

Being defined and based on an equation, "Linear Algebra" thus is a natural part of "Algebra". Clearly this fundamental equation should serve well as the conceptual core and the beginning of our studies and teaching.

Therefore one of the first tasks in elementary Linear Algebra consists of describing all linear transformations $f: R^n \rightarrow R^m$ as matrix × vector multiplication, where the "standard matrix" AE for f and the matrix × vector product are short-hand notations for the action of the linear map f. This puts matrices firmly at the core of Linear Algebra.

Following our desire for balance, we now balance the conceptual with the concrete in our teaching. For this purpose we introduce row reduction and applications to linear equations. Further balancing is needed, however. Row reduction is mechanically tedious to do by pencil on paper. To learn and understand this algorithm, it suffices to practice it over the integers. Hence each teacher must learn how to construct infinitely many integer test problems. When this relatively complicated algorithm, 'relatively complicated' in relation to the math maturity of sophomores, has been balanced with easy integer arithmetic, students realize quickly that linear transformations (or matrices), row reduction, and linear equations are intimately linked.

The geometry of R^n extends, however, beyond mere vectors: Every linear transformation generates two intrinsic and complementary subspaces: the image and kernel. Any description of the image involves the solvability of a linear system $Ax = b$, while every solution to $Ax = 0$ belongs to the kernel. That is, both types of subspaces can be well understood from linear equations and linear transformations $x \rightarrow Ax$. To continue, we may ask: how large are these two elemental subspaces, how can we describe them. One, the image, is a span, while the kernel obeys a set of defining equations. We can translate between these two generic descriptions of a subspace by using special row reduction schemes, and thus we are back to computations.

Subspaces lead to the "king chapter" of any elementary Linear Algebra course, to linear (in) dependence, basis, and dimension. The 'classical' and standard first definition of linear independence of column vectors u_i:

$$\sum_i \alpha_i u_i = 0 \Rightarrow all\ \alpha_i = 0$$

should, however, come only after a transformation based one. We rather introduce linear (in) dependence based on the vector–matrix identification:

$$k \text{ vectors } u_i \in \mathbb{R}^n \quad \longleftrightarrow \quad U = \begin{pmatrix} | & & | \\ u_1 & \dots & u_k \\ | & & | \end{pmatrix}_{nk}$$

the matrix with columns $u_i \in \mathrm{R}^n$.

Now we study the linear transformation $x \in \mathrm{R}^k \mapsto Ux \in \mathrm{R}^n$: How large is the image of R^k under the mapping by U? What vectors generate im(U) = span $\{u_1 \dots u_k\}$?

A row reduction R of U shows that any column vector u_i without a corresponding pivot in R is a linear combination, i.e. linearly dependent of the previous columns u_j for $j < i$.

Hence our preferred first concrete linear independence definition:

A set of vectors $u_1 \dots u_k \in \mathrm{R}^n$ is linearly dependent ⇔ a row echelon form of U_{nk} has less than k pivots. It is linearly independent ⇔ a row echelon form of U_{nk} has k pivots. By applying the unique solvability criterion of linear systems (no free columns) we obtain the 'classical' linear (in)dependence condition. This gives us a dual insight: computing a row echelon form R of the column vector matrix U decides linear independence among the ui practically, while the 'classical' definition helps in abstract settings such as in proofs.

In the same vein we develop a dual definition for the concept of a 'basis', namely: a basis of a (sub)space is defined in two equivalent ways as

a) a maximally linearly independent set of vectors in that (sub)space, or
b) a minimal spanning set of vectors for the (sub)space.
c) Knowing 'basis' leads us to study "basis change". Here we again identify a given basis U = $\{u_1 \dots u_n\}$ of R^n with the column vector matrix

$$U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}_{nn} .$$

For the standard unit vector basis E: = $\{e_1 \dots e_n\}$, a point $x = \sum_i x_i e_i \in \mathrm{R}_n$ has the standard E-basis coordinate vector

$$x = x_{\mathscr{E}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For a basis U = $\{u_1 \dots u_n\}$ of R^n, the point $x = \sum_i \beta_i u_i \in \mathrm{R}^n$ has the U coordinate vector

$$x_{\mathscr{U}} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \quad \text{with } x = Ux_{\mathscr{U}}.$$

For any third basis V = $\{v_1 \dots v_n\}$ we likewise have $x = x_E = V_{x\mathrm{V}}$. Thus

$$Vx_V = x = Ux_U, \text{ or } x_V = V^{-1}Ux_U \text{ and } x_U = U^{-1}Vx_V$$

This points to a practical row reduction scheme for finding basis change matrices $X_{V \leftarrow U}$: = $V^{-1}U$. To be able to compute these easily by hand, teachers must be made familiar with generating unimodular integer matrices.

Everything mentioned and practiced so far ties together when we represent a given linear transformation f: $\mathrm{R}^n \to \mathrm{R}^n$, or its standard matrix representation $A = A_E$, with respect to another basis U. The U basis representation A_U of f maps U-coordinate vectors to U-coordinate vectors, while A_E maps standard E vectors to standard E vectors. A_s $x_E = Ux_U$ and $y_U = U^{-1}y_E$ for all x and $y \in \mathrm{R}^n$, we note that $A_E x_E = A_E U x_U$ and thus "A_U" $x_U = U^{-1}A_E U x_U$ describes the linear transformation completely in terms of U-coordinate vectors. This interprets a basis change as matrix similarity. It links matrix simplification, and specifically diagonalizability, to the notion of an eigenvector and eigenvalue: if $A_E u_i = \lambda_i u_i$ for n linearly independent vectors u_i, then for the column vector matrix

$$U = \begin{pmatrix} | & & | \\ u_1 & \dots & u_n \\ | & & | \end{pmatrix},$$

the matrix $A_U = U^{-1}A_E U$ = diag(λ_i) has a most simple diagonal appearance.

All eigenvalues of A can be found from the least common multiple of the set of all vanishing polynomials $\{p_{xi}$ (A)$\}$ for a basis $\{x_1 \dots x_n\}$ of R^n. (Of course, with hand-computations, we generally practise this only for n 7 4 or 5.) The vector iteration approach leads immediately to the strong version of the Cayley–Hamilton theorem. Vector iteration prepares students naturally for invariant subspaces and the Jordan Normal Form, all in a first course.

For more specific applications we turn to orthogonality. This can be explained by using the stable modified Gram-Schmidt process in analogy to Gaussian elimination, rather than by using its unstable 'classical' variant. (As teachers we need to be above board and not clutter our student's perception with obsolete algorithms that may take years to correctly dismiss.) Most simply said, the process of orthogonalizing k row vectors $u_1 \dots u_k \in \mathrm{R}_n$ in levels viamodified Gram–Schmidt is analogous to row reducing the matrix

$$A = \begin{pmatrix} - & u_1 & - \\ & \vdots & \\ & u_k & - \end{pmatrix}_{kn}$$

via Gaussian elimination.

If the complete row reduction of A to row echelon form R can proceed without any swaps, the first level sweep of row reduction creates row equivalent rows $\tilde{u}_j$ for $j = 2 \dots k$ in A's update that lie in the coordinate plane span $\{e_2 \dots e_n\} \subset \mathrm{R}^n$. Geometrically, the row reduction of one row u_j via the pivot $u_{11} \neq 0$ of row u_1 projects u_j onto span $\{e_2 \dots e_n\}$. On the next level, Gauss uses $\tilde{u}_2$ to update each of the updated rows $\tilde{u}_j$ span $\{e_2 \dots e_n\}$ to lie in span$\{e_3 \dots e_n\}$ of R^n for $j = 3 \dots k$, etc.

That is to create a row echelon form R for A via Gauss we first alter $n - 1$ row vectors in A via A's first row u_1, then we update the trailing and updated $n - 2$ rows of A via its second updated row, etc. (Fig. 1).

The modified Gram–Schmidt algorithm also updates the row vectors $u_1 \dots u_k$ of A in levels, but with an eye on orthogonality rather than on zero leading coefficients (Fig. 2). In modified Gram–Schmidt we first replace each u_j for $j = 2 \dots k$ by a vector $v_j \in \text{span}\{u_1, u_j\}$ that lies in the $n - 1$ dimensional subspace

$$u_1^{\perp} = \{v \in \mathbb{R}^n | v \cdot u_1 = 0\}.$$

Then we update the newly computed vectors $v_3 \dots v_n$ to lie in span $\{v_2, v_j\}$ and in $v_2^{\perp}$ for $j = 3 \dots k$, etc., until we normalize. Further topics such as symmetric and normal matrices, the singular value decomposition, and the Jordan normal form may lie beyond the reach of a one-semester elementary Linear Algebra course. However, they can now be treated easily by matrix theoretic means such as via the Schur Normal form and vector iteration.

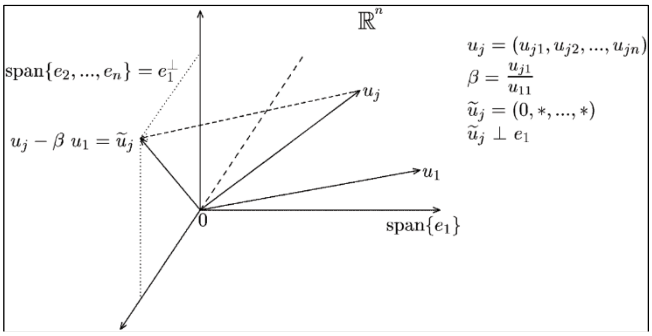

Fig 1: Gaussian elimination

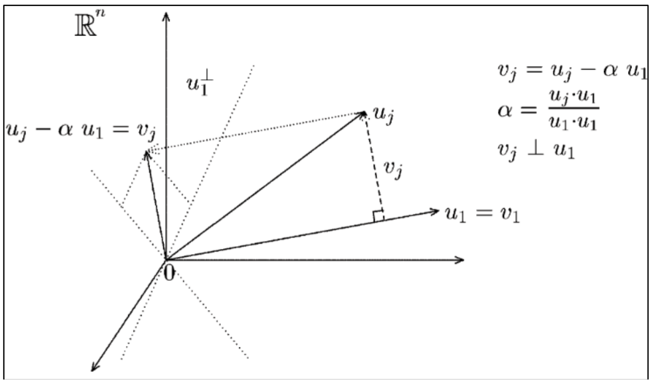

Fig 2: Modified Gram-Schmidt

Results

Rethinking and retooling how to teach elementary Linear Algebra was Emily Haynsworth's desire when I joined her at Auburn. May be I could try to reshape our "miserable 266" introductory course, she suggested in 1982.

The above reformulation of how to teach Linear Algebra from a linear algebraic, unified, balanced, and conceptual viewpoint took 18 years to realize. It has resulted in the textbook "Transform Linear Algebra".

Successes. By following a well-balanced conceptual approach such as ours, students gain math maturity and teachers find satisfaction in their teaching, i.e., no more 'miserable 266' classes.

The approach emphasizes the value of concepts and first principles, making problem understanding and problem solving easy and possible, perhaps for the first time in a student's math career. The entire course becomes a self-validating experience for students and teachers.

Conclusion

Linear Algebra has a high level of internal structure. These inner forces drive our chosen sequence of subjects and determine our depth of conceptualization. Linear transformations can be used to act as the fundamental concept and basis for the whole course. When the structure is exposed and real world applications are solved through conceptual understanding, we serve the students well in their intellectual and personal maturation. When we teach from examples, students tend to become disoriented and confused. They often cannot retain concepts long enough to be able to apply them later.

This approach is difficult for and exposes the 'thinking impaired', 'cannot read', 'no time to study' students. Such learning disorders—around 8–15% of the students now suffer from them in any of my typical undergraduate math courses—become very obvious during a transform based Linear Algebra course.

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