ISSN Print: 2394-7500
ISSN Online: 2394-5869
Impact Factor: 5.2
IJAR 2019; 5(12): 428-430
www.allresearchjournal.com
Received: 05-10-2019
Accepted: 19-11-2019

Dr. Gaurav Kumar
Associate Professor,
Department of Mathematics,
N.A.S. College, Meerut, Uttar Pradesh, India

Correspondence Author:
Dr. Gaurav Kumar
Associate Professor,
Department of Mathematics,
N.A.S. College, Meerut, Uttar Pradesh, India

On lie symmetry solution of first order homogeneous ordinary differential equation

Dr. Gaurav Kumar

Abstract
Ordinary Differentia Equations (ODE) play an important role in various fields including weather forecasting, growth models, decay of radioactive substance among others. ODE are solved by integration and reducing their order. This reduction takes them to first order. Homogeneous differential equations are one of the most used first order differential equations. There are different methods of solving such equations. Symmetry method of solving ODE is given by Sophus Lie. This method is based on finding symmetries of an equation. These symmetries are point transformations under which the original equation remains invariant. In present work, this method is applied to homogeneous differential equation and is explained with the help of an example.

Keywords: Ordinary differential equation, point transformations, lie symmetry, homogeneous differential equation, similarity solutions

1. Introduction

Solving ordinary differential equations was one of the most important topic of Applied Mathematics in 19th century. Several phenomenon of physical science can be described with the help of differential equations. In the course of time several methods have been developed to integrate a differential equation. A Norwegian Mathematician Sophus Lie developed theory of transformations under which the differential equation remains invariant. He called these transformations as point transformations. His theory was based on Galoi's theory of groups as these point transformations form a group called group of transformations. Using these transformations, the order of a differential equation can be reduced so as to make it integrable. In present work, Lie symmetry method is explained for first order ordinary differential equation and then the method is applied to homogeneous differential equation so as to find its point symmetries and its general solution.

2. Related Work

Group invariant solutions of differential equations were studied by Olver and Rosenau (1987) [9]. Dresner (1988) [6] dealt with the similarity solutions of second-order partial differential equations in one dependent and two independent variables that are invariant to a one-parameter family of one-parameter groups. It is shown that this ordinary differential equation is itself invariant to a group of affirm transformations. Bluman (1990) [2]. Found invariant solutions of an ordinary differential equation by solving an algebraic equation which derived from the given ODE and the infinitesimals of an admitted Lie group of transformations. Clarkson and Olver (1996) [9]. Reduced order of an ordinary differential equation by three using a symmetric group and recovered the solution of the reduced equation using a pair of quadratures. Leach *et al*. (2001) [1]. Discussed complete symmetry groups of ordinary differential equations and their integrals. Burde (2002) [3] used expanded transformations to the concept of similarity reductions of partial differential equations. The expanded similarity reductions of differential equations may be used as a tool for finding changes of variables, which convert the original PDE into another simpler PDE. Moitsheki (2007) [8] used Lie group techniques to construct invariant solutions for some cases of concentration and a zero source term to analyze a flow with Contaminated -Modified viscosity. Martinot (2014) [7] explored some applications of the symmetries inherent to ordinary differential equations. Carinena *et al*. (2014) [4] analyzed a class of Lie Systems on Dirac Manifolds. These manifolds are called Dirac-Lie system and these are associated with Dirac-Lie Hamiltonians. Paliathanasis and Leach (2016) [10].

Discussed the symmetry analysis and singularity analysis methods for the study of the integrability of nonlinear ordinary differential equations. Hasan and Abd (2016) presented the cases of symmetry transformations along with the particular case involving an integrating factor by using a coordinate transformation which takes a system of ordinary differential equations with no obvious solution to system of integrables. Chatibi *et al*. (2018) [5] constructed a proper extension of the classical prolongation formula of Lie point transformations for conformable derivative.

3. Method Adopted

In this section, Lie symmetry method for first order ordinary differential equation is explained in brief. Consider the general form of first order ordinary differential equation:

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

Where x and y are independent and dependent variables respectively.
To solve (1), let us consider one parameter family of Lie group

$$\bar{x} = f_1(x, y; \varepsilon),\ \bar{y} = f_2(x, y; \varepsilon). \tag{2}$$

If equation (1) is invariant under transformation (2), then –

$$\frac{d\bar{y}}{d\bar{x}} = f(\bar{x}, \bar{y}), \tag{3}$$

Which gives

$$\frac{f_{2x} + f_{2y} f(x,y)}{f_{1x} + f_{1y} f(x,y)} = f\left(f_1(x, y, \varepsilon), f_2(x, y, \varepsilon)\right). \tag{4}$$

To solve (4), Lie considered extended form of Lie group called infinitesimal transformations. The infinitesimals to transform equation (1) are solutions of Lie's invariance condition given by –

$$\eta_x + (\eta_y - \eta_x) f - \xi_y f^2 = \xi f_x + \eta f_y \tag{5}$$

Solving equation (5) we shall find infinitesimals ξ and η.
Using these infinitesimals, the given equation is transformed to canonical coordinates (r,s) by –

$$r_x \xi + r_y \eta = 0 \tag{6}$$

$$s_x \xi + s_y \eta = 1 \tag{7}$$

The equations (6) & (7) are then solved by method of characteristics to find r and s. Then we use –

$$\frac{ds}{dr} = \frac{s_x + s_y \dfrac{dy}{dx}}{r_x + r_y \dfrac{dy}{dx}} \tag{8}$$

Equation (8) transforms equation (1) to separable type of equation in new co-ordinates (r,s). The equation (8) is then solved by usual method to find the general solution of equation (1).

4. Example

In this section, symmetries of a first order homogeneous differential equation are used to find its general solution. Consider first order homogeneous differential equation –

$$\frac{dy}{dx} = \frac{y^2 + x^2}{2xy} \tag{9}$$

Comparing (9) with (1), we have –

$$f(x, y) = \frac{y^2 + x^2}{2xy} \tag{10}$$

Let us consider one parameter family of Lie group as –

$$(\bar{x}, \bar{y}) = (e^{\lambda} x, e^{\lambda} y) \tag{11}$$

Here λ is a parameter.
From (1), we get –

$$\frac{d(e^{\lambda} y)}{d(e^{\lambda} x)} = \frac{(e^{2\lambda} y^2 + e^{2\lambda} x^2)}{(2e^{\lambda} x e^{\lambda} y)}$$

$$\frac{e^{\lambda} dy}{e^{\lambda} dx} = \frac{e^{2\lambda}(y^2 + x^2)}{(2e^{2\lambda} xy)}$$

which gives –

$$\frac{dy}{dx} = \frac{(y^2 + x^2)}{(2xy)} \tag{12}$$

Therefore (9) is invariant under the transformation (11).
Thus we have –

$$f_1(x, y, \lambda) = e^{\lambda x} \tag{13}$$

$$f_2(x, y, \lambda) = e^{\lambda y} \tag{14}$$

Now infinitesimals are given by–

$$\xi = \left(\frac{df_1}{d\lambda} \right)_{\lambda = 0} = x \tag{15}$$

$$\eta = \left(\frac{df_2}{d\lambda} \right)_{\lambda = 0} = y \tag{16}$$

From (6) and (7), equations in canonical coordinates are –

$$r_x x + r_y y = 0 \tag{17}$$

$$s_x x + s_y y = 1 \tag{18}$$

Lagrange's auxiliary equations for (17) are –

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dr}{0} \quad (19)$$

We get –

$$r = c_1 \quad (20)$$

and $\frac{y}{x} = c_2$ (21)

where c_1 and c_2 are arbitrary constants.
Therefore –

$$r = \frac{y}{x} \quad (22)$$

Lagrange's auxiliary equations for (18) are –

$$\frac{dx}{x} = \frac{dy}{y} = \frac{ds}{1} \quad (23)$$

Two independent solutions of (23) are –

and $s = \log c_3 x$ and $\frac{y}{x} = c_4$ (24)

where c_3 and c_4 are arbitrary constants.
Now from (8), we get –

$$\frac{ds}{dr} = \frac{\frac{1}{x} + 0 \times \frac{dy}{dx}}{\frac{-y}{x^2} + \frac{1}{x}\frac{dy}{dx}} \quad (25)$$

On simplifying (25), we get –

$$\frac{ds}{dr} = \frac{2xy}{x^2 - y^2} \quad (26)$$

Further from (26), we have –

$$\frac{ds}{dr} = \frac{2xy}{x^2(1-\frac{y^2}{x^2})} = \frac{2\frac{y}{x}}{(1-\frac{y^2}{x^2})}$$

$$\Rightarrow \frac{ds}{dr} = \frac{2r}{(1-r^2)}$$

$$\Rightarrow ds = \frac{2r}{(1-r^2)} dr$$

$$\Rightarrow s = -\log(1-r^2) + \log k$$

$$\Rightarrow s = \log \frac{k}{(1-r^2)}$$

$$\Rightarrow \log x = \log \frac{k}{(1-r^2)}$$

$$\Rightarrow x = \frac{k}{(1-r^2)}$$

$$\Rightarrow k = x(1-\frac{y^2}{x^2})$$

$$\Rightarrow x^2 - y^2 = kx \quad (27)$$

where k is an arbitrary constant.
Equation (27) is required general solution of (9).

5. Conclusion

Lie symmetry method is a robust method to solve ordinary differential equation. In this work, first order differential equation is solved using this method. This method provides exact solution to first order homogeneous ODE. Lie method is equally applicable for higher order differential equations as well. Lie symmetries transform higher order ODE to lower order ODE so as to reduce their complexity. This method can also be used for non-linear ODE. Lie symmetry method is an algorithm based method and hence it can be applied using software like MATLAB, SCILAB and Mathematica among others.

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