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Essentially small Quasi-Dedekind modules and related concepts

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Abstract
Let R be a ring with identity and S be a unitary left Module over R. In this paper, we give many connections between essentially small quasi-Dedekind (small quasi-Dedekind) modules and other modules such that Baer modules, retractable modules and small compressible modules.

Keywords: small quasi-Dedekind, essentially small quasi-Dedekind

Introduction
A submodule A of an R-module M is called small in M ($A \ll M$) if whenever a submodule B of M with M = A + B implies B = M [1]. An R-submodule N of an R-module M is called essentially small ($N \ll_e M$), if for every nonzero small submodule K of M, $K \cap N \neq 0$. Equivalently, for each $0 \neq x \in M$, there exists $0 \neq r \in R$ such that $0 \neq rx \in N$. Asubmodule N of an R-module M is called small invertible if $N^{-1}N = M$,where $N^{-1} = \{x \in R_T : xN \ll M\}$ and R_T is the localization of R at T in the usual sence, $T = \{s \in S: sm = 0$ for some $m \in M$, then $m = 0\}$, where S is the set of all nonzero divisors of R. An R-module M is called small quasi-Dedekind, if every nonzero R submodule N of M is small quasi-invertible; that is $Hom(M/N, M) = 0$, for all $0 \neq N \ll M$. A ring R is small quasi-Dedekind if R is a small quasi-Dedekind R-module. An R-module M is called essentially small quasi-Dedekind if $Hom(M/N, M) = 0$ for all $N \ll_e M$. [2]. A ring R is essentially small quasi-Dedekind if R is an essentially small quasi-Dedekind R-module.
An R-module M is Baer if, for all $N \leq M$, $L_S(N) = Se$, with $e^2 = e \in S = End_R(M)$. Equivalently, M is Baer if, for all ideal $I \leq_S S$, $r_M(I) = eM$ with $e^2 = e \in S = End_R(M)$, [3,P.10].

Proposition 1 Every Baer R-module is an essentially small quasi-Dedekind (K-nonsingular) R-module.

Proof: Suppose that M is a Baer R-module, let $f \in End_R(M)$, $f \neq 0$. To prove that Kerf $\not\ll_e$ M. If Kerf = 0 then M is a small quasi-Dedekind R-module, so it is an essentially small quasi-Dedekind R-module. If $Kerf \neq 0$, since M is Baer, then by [4, Th.1.5] $Kerf \leq^{\oplus} M$. Thus Kerf $\not\ll_e$ M. Thus M is an essentially small quasi-Dedekind R-module.
The converse of Prop.1 is not true in general, as the following example shows.

Example 2 It is well-known that by [3,Ex 2.4.2], $Z^N = Z \oplus Z \oplus Z \ldots$ as a Z-module is not Baer. But Z is an essentially small quasi Dedekind Z-module; that is Z is an essentially small quasi-Dedekind relative to Z. Then by [5, Th. 4], $Z^N = Z \oplus Z \oplus Z \ldots$ is an essentially small quasi-Dedekind Z-module.

Corollary 3 If M is a Baer R-module, then $End_R(M)$ is an essentially small quasi-Dedekind ring.

Proof: Since M is a Baer R-module, so by [3, Th. 4.1.1], $End_R(M)$ is a Baer ring. Thus by Prop.1, $End_R(M)$ is an essentially small quasi-Dedekind ring.

An R-module M is retractable if, $Hom(M, N) \neq 0$, for all $0 \neq N \leq M$, [3, P. 43].

Corollary 4 Let M be a retractable R-module. If $End_R(M)$ is a Baer ring, then M is an essentially small quasi-Dedekind R-module.

Proof: Suppose that $End_R(M)$ is a Baer ring, and since M is a retractable R-module, so by [3,Prop 4.1.4], M is a Baer R-module. Thus by Prop.1, M is an essentially small quasi-Dedekind R-module.

Proposition 5 Let M be a retractable R-module. If M is an essentially small quasi-Dedekind R-module, then S = $End_R(M)$ is a right nonsingular ring and hence essentially small quasi-Dedekind.

Proof: Suppose that M is an essentially small quasi-Dedekind R-module. To prove that S_s is a nonsingular ring, where S = $End_R(M)$, we must prove $\{\phi \in S : r_S(\phi) \ll_e S_s\} = 0$. Let $\phi \in S$ such that $r_S(\phi) \ll_e S_s$. If $\phi = 0$, no thing to prove. If $\phi \neq 0$, then $Ker\ \phi \ll_e M$. Hence there exists $0 \neq N \leq M$, N is a relative complement to $Ker\ \phi$, and so $N \cap Ker\ \phi = 0$. By retractability of M, there exists $\psi : M \longrightarrow N$, $\psi \neq 0$. Consider the following: $M \xrightarrow{\psi} N \xrightarrow{i} M \xrightarrow{\phi} M$, where i is the inclusion mapping. $\phi o i o \psi \neq 0$, to show thi: assume $\phi o i o \psi = 0$. Since $\psi \neq 0$, there exists $m \in M$ such that $\psi(m) = n \neq 0$, $n \in N$, hence $0 = \phi o i o \psi(m) = (\phi o i)(n) = \phi(n)$ which implies $n \in Ker\ \phi$, hence $n \in N \cap Ker\ \phi = 0$ which is a contradiction. Thus $\phi o i o \psi \neq 0$, so $\phi o \psi \neq 0$, implies $\psi \notin r_S(\phi)$, then $0 \neq \psi s \not\subseteq r_S(\phi)$. We claim that $\psi s \cap r_S(\phi) = 0$. Suppose that, there exists $g \in \psi s$ and $g \in r_S(\phi)$, then $g = \psi o h$ and $\phi o g = 0$ for some $g, h \in S = End_R(M)$, so $g(M) = \psi o h(M) \subseteq \psi(M) \subseteq N$ which implies $g(M) \subseteq N$, also $\phi o g(M) = 0$, then $\phi(g(M)) = 0$ implies $g(M)) \subseteq Ker\ \phi$ .Thus $g(M)) \subseteq N \cap Ker\ \phi = 0$; that is $g = 0$. But this contradicts the essentiality of $r_S(\phi)$. Therefore $Ker\ \phi \ll_e$ M and hence $\phi = 0$, since M is an essentially small quasi-Dedekind R-module. Thus S = $End_R(M)$ is a right nonsingular ring and hence essentially small quasi-Dedekind.

We prove the following proposition:

Proposition 6 Let M be a uniform R-module. If M is an essentially small quasi-Dedekind R-module then M is a Baer R-module.

Proof: Since M is a uniform R-module, so by [6, Prop. 2.1.1], M is an extending R-module. But M is an essentially small quasi-Dedekind extending R-module, implies M is a Baer R-module, by [3, lemma 2.2.4].

Proposition 7 Let M be a uniform R-module. If M is a Baer R-module then M is a small quasi-Dedekind R-module.

Proof: Since M is a uniform R-module, so by [6, Prop. 2.1.1], M is an indecomposable R-module. But M is a Baer and indecomposable R-module, implies M is a small quasi-Dedekind R-module, by [3, Th 2.3.5].

To give the next result, first we prove the following lemma.

Lemma 8 Let M is a uniform R-module, then E(M) is a uniform R-module.

Proof: Let $U \leq E(M)$, $U \neq 0$. To prove $U \cap W \neq 0$ for all $0 \neq W \leq E(M)$. Since $M \leq_e E(M)$, then $U \cap M \neq 0$, $W \cap M \neq 0$. But since $U \cap M \leq M$, $W \cap M \leq M$ and M is uniform, so that $(U \cap M) \cap (W \cap M) \neq 0$. This implies $(U \cap W) \cap M \neq 0$, hence $U \cap W \neq 0$. Thus E(M) is a uniform R-module.

Corollary 9 Let M be an R-module and $ann_R(M) = ann_R(E(M))$. Then M is a uniform small quasi-Dedekind R-module if and only if E(M) is a uniform small quasi-Dedekind R-module.

Proposition 10 Let M be a uniform R-module with $ann_R(M) = ann_R(E(M))$. The following statements are equivalent:

1) If E(M) is an essentially small quasi-Dedekind R-module then E(M) is a Baer R-module.
2) If E(M) is a Baer R-module then E(M) is a small quasi-Dedekind.
3) E(M) is a small quasi-Dedekind R-module if and only if M is a small quasi-Dedekind R-module.

Proof: (1) It follows by Lemma8 and Prop.6.
(2) It follows by Lemma8 and Prop7.
(3) It follows by Lemma 8 and Coro9.

An R-module M is called small compressible if M can be embedded in each of its non-zero small submodule. Equivalently, M is small compressible if there exists a monomorphism from M into N whenever $0 \neq N \ll M$ [7].

A ring R is called small compressible if the R-module R is small compressible. That is R can be embedded in any of its non-zero small ideal.

An R-module M is small monoform if, for each $N \ll M$ and for each $f \in Hom(N, M)$, $f \neq 0$ implies Kerf = 0 [8].

An R-module M is said to be an small prime if $ann_R M = ann_R N$ for each $N \ll M$ [9].

An R-module M is called essentially small prime if, $ann_R(M) = ann_R(N)$ for all $N \ll_e M$ [9].

An R-module S is small polyform whether, every small essential Submodule of S is rational in S [10].

A nonzero R-module M is essentially small compressible if, M embedded in each essential small submodule. It is clear that every small compressible module is an essentially small compressible module.

Proposition 11 Every small compressible R-module is a small monoform R-module, and hence a small quasi-Dedekind R-module.

The converse of Prop.11, is not true in general, for example: The Z-module Q is uniform and small prime, hence it is small monoform [10, Prop.13], but it is not small compressible, since $Hom(Q, Z) = 0$; that is Q can not be embedded in Z.

The converse of Prop.11 holds if, M is finitely generated.

Proposition 12 Let M be a finitely generated R-module. Then M is small compressible if and only if M is small monoform.

Proof: $\Rightarrow$) It is clear by Prop.11.

$\Leftarrow$) By [11, Th 2.3], M is a uniform small prime R-module. But M is finitely generated, so by [11, Lemma 1.9], M is small compressible.

The condition M is finitely generated can not be dropped from Prop.12. For example: The Z-module Q is small monoform, but it is not small compressible. In fact Q is not finitely generated.

Proposition 13 Let M be a finitely generated R-module.Then M is small compressible if and only if M is small monoform.

Corollary 14 Let M be a finitely generated R-module. The following statements are equivalent:

1) M is a small monoform R-module.
2) M is a uniform small prime (uniform essentially small prime) R-module.
3) M is a uniform small quasi-Dedekind (uniform essentially small quasi-Dedekind) R-module.
4) M is a small compressible R-module.

Proof: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): It follows by [10, Prop.13].

(1) $\Leftrightarrow$ (4): It follows by Prop.13.

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