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On ternary quadratic equation with three unknowns $x^2 + xy + y^2 = 36z^2$

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Abstract
The Ternary quadratic Diophantine equation is given by $x^2 + xy + y^2 = 36z^2$ is analyzed for its methods of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: Ternary, quadratic, integral solutions, polygonal numbers

1. Introduction
The ternary quadratic Diophantine equation offers an unlimited field for research because of their variety [1, 3]. For an extensive review of various problems, one may refer [1, 20]. This communication concerns with yet another interesting ternary quadratic equation $x^2 + xy + y^2 = 36z^2$ for determining its infinitely many non-zero distinct integral solutions. Also a few interesting relations among the solutions have been exhibited.

Notations
1. $T_{3,n}$ - Polygonal number of rank n with size m
2. P_n^3 - Tetrahedral number of rank n
3. P_n^4 - Square pyramidal number of rank n
4. P_n^5 - Pentagonal pyramidal number of rank n
5. Gno_p - Gnomonic number of rank p.
6. FN_4^a - Four-dimensional figurate number of rank n.
7. $T_{3,n}$ - Heptagonal number of rank n with size m

2. Method of Analysis
The Ternary quadratic Diophantine equation to be solved for its non-zero distinct integral solution is

$$x^2 + xy + y^3 = 36\,z^2 \qquad \text{--- (1)}$$

On substitution of linear transformation ($u \neq v \neq 0$)

$$x = u + 3v, y = u - 3v \qquad \text{--- (2)}$$

in (1) leads to

$$u^2 + 3v^2 = 12z^2 \qquad \text{--- (3)}$$

Now we solve equation (3) through different methods and thus obtain different patterns of solutions to (1)

Method 1
Imagine that

$$Z = (a, b) = a^2 + 3b^2 \qquad \text{--- (4)}$$

Where a and b are non-zero distinct integers
Mark 12 as

$$12 = \frac{(3n+ni\sqrt{3})(3n-ni\sqrt{3})}{n^2} \quad \text{--- (5)}$$

using (4) and (5) in equation (3) and applying the way of factorization, classify,

$$u + i\sqrt{3}v = \frac{1}{n}\left\{\left(3n + ni\sqrt{3}\right)(\alpha + i\sqrt{3}b^2)\right\}$$

Equating the real and imaginary parts, are get

$$\left.\begin{aligned} u &= 3a^2 - 6ab - 9b^2 \\ v &= a^2 + 6ab - 3b^2 \end{aligned}\right\} \quad \text{--- (6)}$$

Substituting (6) in (2) the values of x and y are given by

$$\left.\begin{aligned} x &= x\,(a,b) = 6a^2 + 12\,ab - 18b^2 \\ y &= y\,(a,b) = -24\,ab \\ z &= z\,(a,b) = a^2 + 3\,b^2 \end{aligned}\right\} \quad \text{--- (7)}$$

Thus the equation (7) represent the non-zero integral solutions to (1)
A few interesting properties observed are as follows:

Note
Instead of (2) using the transformation $x = u - 3v, y = u + 3v$ in (1), we get again (3) only, thus the integer solutions of (1) are obtained as

$$\left.\begin{aligned} x &= x(a,b) = -24\alpha b \\ y &= y(a,b) = 6a^2 + 12ab - 18b^2 \\ z &= z\,(a,b) = a^2 + 3b^2 \end{aligned}\right\} \quad \text{--- (8)}$$

Thus the equation (8) represent the non-zero integral solutions to (1).
A few interesting properties observed are as follows:
- $x\,(a, 2a^2 + 1) + 864\,FN_a^4 - 12\,Cc_a + 17\,Star_a \equiv 47\,(mod\,126)$
- $2x\,(2a, 1) + y(2a, 1) - 48\,T_{4,a} + 36 = 0$
- $y\,(3a, 1) + 3z(3a, 1) - 27\,Ct_{2,a} \equiv 18\,(mod\,99)$
- $x\,(5a, 1) - 6z(5a, 1) - 30\,Gno_a = 15$
- $2x\,(10a, 1) + y(10a, 1) - 1200\,T_{4,a} + 36 = 0$
- $2x\,(a, a(a+1)) + y\,(a, a\,(a+1)) + 9\,(Pro_a * CS_a) + 6\,T_{13,a} + 9Gno_a + 27 = 0$
- $y\left(a, a+1\right) + 7z\,(a, a+1) - 4\,Ct_{2,a} \equiv 3\,(mod\,14)$

Each of the following expression represents a nasty numbers
- $x\,(a,a) + y(a,a)$
- $y\,(-a,a)$
- $3z\,(a,a)$
- $2x\,(a,a) + 4y(a,a)$
- $4z\,(a,a)$ is a perfect square

Method 2
Rewrite equation (3) can be written as

$$u^2 + 3v^2 = 12z^2 * 1 \quad \text{--- (9)}$$

Mark 1 as

$$1 = \frac{(n+ni\sqrt{3})(n-ni\sqrt{3})}{(2n)^2} \quad \text{--- (10)}$$

Using (4), (5) and (10) in equation (9) and applying the way of factorization, classify,

$$u + i\sqrt{3}v = \frac{1}{2n^2}\left\{\left(3n + ni\sqrt{3}\right)\left(n + ni\sqrt{3}\right)\right\}\left(\alpha + i\sqrt{3}b\right)^2 \quad \text{--- (11)}$$

Equating the real and imaginary parts, we get

$u = -12ab$

$v = 2a^2 - 6b^2$

Hence in view of (2), the values of x and y are given by

$$\left.\begin{aligned} x &= x(a,b) = 6a^2 - 12ab - 18b^2 \\ y &= y(a,b) = -6a^2 + 18b^2 - 12ab \\ z &= z(a,b) = a^2 + 3b^2 \end{aligned}\right\} \quad \text{--- (12)}$$

Thus the equation (12) represent the non-zero integral solutions to (1)
A few interesting properties observed are as follows:

- $x(a,1) + y(a,1) - 2T_{3,a} \equiv 3 \ (mod\ 25)$
- $x(1,b) + y(1,b) + z(1,b) - T_{8,a} \equiv 1 \ (mod\ 22)$
- $y(a,1) + 6z(a,1) \equiv 0 \ (mod\ 12)$
- $x(a,(a+1)(a+2)) + y(a,(a+1)(a+2)) + 144\, P_a^5 = 0$
- $x(a,a(a+1)) + 6z(a,a(a+1)) - 12\ Cp_a^6 = 0$
- $y(a,a+1) - 6z(a,a+1) + 7\,T_{4,a} + 24T_{3,a} = 0$

Each of the following expression represents a nasty numbers

- $x(a,a+1) + y(a,a+1) + z(a,a+1) + 8\ T_{7,a} - 3Gno_a$
- $x(a,a) - y(a,a)$
- $6z(a,a)$
- $y(a,a) + 6z(a,a)$
- $x(a,a) + y(a,a) - z(a,a)$

Method 3
Equation (3) can be written as

$$u^2 + 3v^2 = 4 * 3z^2 \quad \text{--- (13)}$$

Mark 4 and 3 as

$$\left.\begin{aligned} 4 &= \frac{(n+ni\sqrt{3})(n-ni\sqrt{3})}{n^2} \\ 3 &= \frac{(3n+ni\sqrt{3})(3n-ni\sqrt{3})}{4n^2} \end{aligned}\right\} \quad \text{--- (14)}$$

Using 4 and (14) in (13) and applying the method of factorization, classify,

$$u + i\sqrt{3}v = (n + ni\sqrt{3})(3n + ni\sqrt{3})(\alpha + i\sqrt{3}b)^2$$

Following a similar procedure as in Method – II and the corresponding solution of (1) are same as in Method – II.

Method 4:
Rewrite equation (3) as

$$u^2 + 3v^2 = 4 * 3z^2 * 1 \quad \text{--- (15)}$$

Using equation (4), (10) and (14) in (15) and applying the method of factorization, classify,

$$u + i\sqrt{3}v = \frac{1}{4n^3}\{(n + ni\sqrt{3})(3n + ni\sqrt{3})(n + ni\sqrt{3})(a + i\sqrt{3}b)^2\} \quad \text{--- (16)}$$

Equating the real and imaginary parts, we get

$u = -3a^2 - 6ab + 9b^2$
$v = a^2 - 6ab - 3b^2$

Hence in the view of (2), the values of x and y are given by

$x = x\,(a,b) = -24ab$
$y = y\,(a,b) = -6a^2 + 12ab + 18b^2$ --- (17)
$z = z\,(a,b) = a^2 + 3b^2$

Thus the equation (17) represent the non-zero integral solutions to (1)
A few interesting properties observed are as follows:

- $x\,(a, 2a^2 + 1) + 5z\,(a, 2a^2 + 1) - 1440PT_a - 816\,P_a^5 - 187\,T_{4,a} \equiv 0\,(mod\,384)$
- $x\,(3, a) + 2y(3, a) \equiv 0\,(mod\,36)$
- $x\,(3, b) + y(3, b) + 6z(3, b) - 36T_{4,b} + 18\,Gno_b = 18$
- $x\,(4, b(b+1)) + 2y\,(4, b\,(b+1)) - 36\,FN_a^4 - 144P_a^5 + 192 = 0$
- $y\,(b, b(b+1)) + 6z\,(b, b\,(b+1)) - 36\,(Pro_a * T_{4,a}) - 96CP_a^3 = 0$
- $x\,(5b, b+1) + 7z\,(5b, b+1) - 76\,T_{3,b} + 77Gno_b = 56$

- Each of the following expression represents a nasty numbers
  - $x(4, a) + 2y(4, a) - 1\,2z\,(4, a)$
  - $6z\,(a, a)$
  - $x\,(a, a) + y(a, a) + 6z\,(a, a)$
  - $3[y(b, b) + z\,(b, b)]$
  - $x\,(b, b) + y(b, b) + 6\,[4z(b, b)]$
  - $216\,[4z\,(b, b)]$

Method 5
Equation (3) can be written as

$3v^2 - 3z^2 = 9z^2 - u^2$
$3(v+z)(v-z) = (32+u)3z - u)$

Four different choices of solution obtained are as follows:

Choice I

$x = x\,(A,B) = -6A^2 - 12AB + 18B^2$
$y = y\,(A,B) = 24AB$
$z = z\,(A,B) = A^2 + 3B^2$

Choice II

$x = x\,(A,B) = -24AB$
$y = y\,(A,B) = 6A^2 - 12AB - 18B^2$
$z = z\,(A,B) = A^2 + 3B^2$

Choice III

$x = x\,(A,B) = -6A^2 + 12AB - 18B^2$
$y = y\,(A,B) = 6A^2 - 12AB - 18B^2$
$z = z\,(A,B) = -A^2 - 3B^2$

Choice IV

$x = x\,(A,B) = 6A^2 - 12AB - 18B^2$
$y = y\,(A,B) = -6A^2 + 12AB - 18B^2$
$z = z\,(A,B) = -3A^2 - B^2$

3. Conclusion
In this paper we have presented five different methods of non-zero distinct integer solutions of the ternary quadratic equation given by $x^2 + xy + y^2 = 36z^2$. To conclude, one may search for other methods of solutions and their corresponding properties.

4. References
1. Dickson LE. History of Theory of numbers, Chelsea Publishing Company, New York, 1952, 2.
2. Mordell LJ. Diophantine Equations, Academic Press, London, 1969.
3. Andre Weil, Number Theory. An approach through history: from hammurapi to legendre/Andre weil: Boston (Birkhauser Boston, 1983.

4. Nigel Smart P. The algorithimic Resolutions of Diophantine equations, Cambridge University press, 1999.
5. Smith DE. History of mathematics Dover publications, New York. 1953; I(II).
6. Gopalan MA. Note on the Diophantine equation $x^2 + axy + by^2 = z^2$ Acta Ciencia Indica. 2000; XXVIM(2):105-106.
7. Gopalan MA. Note on the Diophantine equation $x^2 + xy + y^2 = 3z^2$, Acta Ciencia Indica. 2000; XXVIM(3):265-266.
8. Gopalan MA, Ganapathy R, Srikanth R. On the Diophantine equation $z^2 = Ax^2 + By^2$, Pure and Applied Mathematical Sciences. 2000; LII(1-2):15-17.
9. Gopalan MA, Anbuselvi R. On Ternary Quadratic Homogeneous Diophantine equation $x^2 + Pxy + y^2 = z^2$, Bulletin of pure and Applied Sciences. 2005; 24E(2):405-408.
10. Gopalan MA, Vidhyalakshmi S, Krishnamoorthy A. Integral solutions Ternary Quadratic $ax^2 + by^2 = c(a + b)z^2$, Bulletin of pure and Applied Sciences. 2005; 24E(2):443-446.
11. Gopalan MA, Vidhyalakshmi S, Devibala. Integral solutions of $ka(x^2 + y^2 + bxy) = 4ka^2z^2$, Bulletin of pure and Applied Sciences. 2006; 25E(2):401-406.
12. Gopalan MA, Vidhyalakshmi S, Devibala. Integral solutions of $7x^2 + 8y^2 = 9z^2$, Pure and Applied Mathematika Sciences. 2007; LXVI(1-2):83-86.
13. Gopalan MA, Vidhyalakshmi S. An observation on $kax^2 + by^2 = cz^2$, Acta Cienica Indica. 2007; XXXIIIM(1):97-99.
14. Gopala MA, Manjusomanath, Vanitha N. Integral solutions of $kxy = m(x + y) = z^2$, Acta Cienica Indica. 2007; XXXIIIM(4):1287-1290.
15. Gopalan MA, Kaliga Rani J. Observation on the Diophantine Equation $y^2 = Dx^2 + y^2$, Impact J Sci Tech. 2008; 2(2):91-95.
16. Gopalan MA, Vijayasankar A. Observations on a Pythagorean Problem, Acta Cienica Indica. 2010; XXXVIM(4):517-520.
17. Anbuselvi R, Kannaki K. On ternary Quadratic Equation $11x^2 + 3y^2 = 14z^2, 2016$; 5(2):65-68.
18. Anbuselvi R, Kannaki K. On ternary Quadratic Equation $x^2 + xy + y^2 = 12z^2$ IJAR. 2016; 2(3):533-535.
19. Anbuselvi R, Kannaki K. On ternary Quadratic Equation $3(x^2 + y^2) - 5xy + x + y + 1 = 15z^3$ IJSR. 2016: 5(9); 42-48.
20. Anbuselvi R, Kannaki K. On ternary Quadratic Diophantine Equation $7(x^2 + y^2) - 13xy + x + y + 1 = 31z^2$ IERJ. 2017: 3(2):52-57.