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The improper integrals and their convergence

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Abstract
Without any shadow of doubt, Improper Integrals are said to be convergent if the limit is finite and that is the value of the improper integral. In other case, improper integral is divergent if the limit does not exist.
So here each integral is defined as a limit if the limit is finite then we say the integral converges, while if the limit is infinite or does not exist, we can say that the integral diverges. Convergence is taken as positive (means we can do the integral; divergence, on the other hand, is somewhat negative that means, we can't do the integral.

Keywords: Convergence, divergence, improper integral, limit

Introduction
Example-1

Find $\int_0^\infty e^{-x}\,dx$. **(if it even converges)**

Solution:

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_a^b = \lim_{b\to\infty} - e^{-b} + e^0 = 0 + 1 = 1$$

Hence, the integral converges and equals 1

Example-2
Evaluate the integral if it even converges,

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$$

Solution: By definition, $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{c}\frac{1}{1+x^2}\,dx + \int_{c}^{\infty}\frac{1}{1+x^2}\,dx$.

Here, we get to pick whatever c wants. e.g. if we assume c= 0.

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \lim_{b\to-\infty}\left[\arctan(x)\right]_b^0 = \lim_{x\to-\infty}\left[\arctan(0) - \arctan(b)\right]$$

$$= 0 - \lim_{b\to-\infty}\arctan(b) = \frac{\pi}{2}$$

In the same way

$$\int_0^{\infty}\frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Therefore, $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Example-3
The P-test

The integral $\int_1^{\infty} \frac{1}{x^P}\,dx$

1. Converges if P > 1
2. Diverges if P ≤ 1

Example-4

$$\int_1^{\infty} \frac{1}{x^{3/2}}\,dx = \lim_{x\to\infty}\left[\frac{2}{x^{1/2}}\right]_1^b = 2\ ,$$

While $\int_1^{\infty} \frac{1}{x^{1/2}}\,dx = \lim_{b\to\infty}\left[2\sqrt{x}\right]_1^b = \lim_{b\to\infty} 2\sqrt{b-2} = \infty$, and

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln(x)\right]_1^b = \lim_{b\to\infty} \ln(b) - 0 = \infty$$

Convergence vs Divergence
In each case, if the limit exists (or if both limits exist), we say the improper integral converges.
If the limit fails to exist or is infinite, the integral diverges, if enter limit fails to exist or is infinite, the integral diverges.

Example-4

Find $\int_0^2 \frac{2x}{x^2-4}\,dx$ (if it converges)

The demoninator of $\frac{2x}{x^2-4}$ is 0 when x = 2.
So the function is not even defined when x = 2
So,

$$\int_0^2 \frac{2x}{x^2-4}\,dx = \lim_{x\to 2}\int_0^c \frac{2x}{x^2-4}\,dx = \lim_{c\to 2-}\left[\ln\ (x^2-4)\right]_0^c$$

$$= \lim_{c\to 2-} \ln\ \left|x^2-4\right| - \ln\ (4) = -\infty$$

so the integral diverges.

Example-5

Find $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$, if it converges

Solution:

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \left[3(x-1)^{\frac{1}{3}}\right]_0^3$$

but it is right : The function f (x) = $\frac{1}{(x-1)^{\frac{2}{3}}}$ is undefined when x = 1, so we need to split the problem into two integrals.

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx$$

Here, two integrals on the right hand side both converge and add up to

$3\left[1+2^{1/3}\right]$, so $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3\left[1+2^{1/3}\right]$

Test for convergence and divergence-

1. If you are smaller than something that converges, then you converge.
2. If you are bigger than something that diverges then you diverge.

Theorem

Let f and g be continuous on [a, ∞] with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\int_a^\infty f(x)\,dx$ **converges if** $\int_a^\infty g(x)\,dx$ **converges.**

2. $\int_a^\infty g(x)\,dx$ **diverges if diverges** $\int_a^\infty f(x)\,dx$

Example-6

(a) $\int_1^\infty e^{-x^2}\,dx$ **(b)** $\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$.

Solution

Both integrals converge.

(a) Note that $0 < e^{-x^2} \le e^{-x}$ for all $x \ge 1$, and from example we come across

$\int_1^\infty e^{-x^2}\,dx = \frac{1}{e}$, so $\int_1^\infty e^{-x^2}\,dx$ converges

(b) $0 \le \sin^2(x) \le 1$ for all x, so $\int_1^\infty \frac{\sin^2(x)}{x^2} \le \frac{1}{x^2}$

for all $x \ge 1$

since $\int_1^\infty \frac{1}{x^2}\,dx$ converges (by p-test), so does $\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$.

Limit comparison test

Theorem

If positive functions f and g are continuous on [a, ∞] and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L,\ 0 < L < \infty,$$

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ **both converge or both diverge together.**

Example-7

Let f (x) = $\frac{1}{\sqrt{x}+1}$; consider

$$\int_{1}^{\infty} \frac{1}{\sqrt{x}+1}\, dx$$

Here does the integral converge or diverge?

Solution:

We note that *f* looks a lot like g (x) = $\frac{1}{\sqrt{x}}$, and

$\int_{1}^{\infty} g(x)\, dx$ diverges by the p-test, further

$$\lim_{x\to\infty} \int \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{x}+1} = 1$$

So, LCT says $\int_{1}^{\infty} \frac{1}{\sqrt{x}+1}\, dx$ diverges

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