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Poisson NHE distribution: Properties and applications

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Abstract
We introduce a new distribution generated by using the Poisson-G-family with parent distribution as NHE distribution named Poisson NHE distribution. Relevant properties and applications exhibited by the proposed distribution have been described to provide a better understanding to the distribution. The model parameters of the proposed distribution are estimated estimation methods including Maximum Likelihood Estimation. The software platform of R was used for the necessary computation. The evaluation of the proposed distribution's goodness of fit performed through its fitting in comparison with some of the other existing life-time models that use a real data set.

Keywords: Poisson-G family, NHE distribution, MLE, R software, estimation methods

Introduction
In most of the literature of probability it is found that the study of survival analysis in various fields of applied statistics and life sciences, the probability distributions often finds its use. In modeling survival data, existing models do not always present a better fit. Hence most of the researchers are paying attention to generalizing classical distributions and investigating their flexibility and applicability. Usually, these new generalized models provide an improved fit as compared to usual classical distributions which are acquired with introduction to baseline distribution one or more additional shape parameter(s).

For a few decades, it is found that the exponential distribution is taken as base distribution to introduce novel distributions. The modifications of the exponential distribution were introduced by different researchers, some of them are, beta exponential (Nadarajah and Kotz, 2006) [25], Gupta and Kundu (2007) [10] have presented the generalized exponential (GE) with some development, Abouammoh & Alshingiti (2009) [1] has introduced the reliability estimation of the generalized inverted exponential distribution, beta GE (Barreto-Souza *et al.*, 2010) [4], Exponential Extension (EE) distribution (Kumar, 2010) [14], KW (Kumaraswamy) exponential (Cordeiro and de Castro, 2011) [8], Nadarajah & Haghighi (2011) [24] have presented an extension of the exponential distribution, gamma EE by (Ristic and Balakrishnan, 2012) [27], Lemonte, A. J. (2013) [16] has introduced a new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Gomez *et al.* (2014) [9] have presented the exponential distribution with a new extension. The exponentiated exponential geometric by (Louzada *et al.*, 2014) [17], Mahdavi & Kundu (2017) [20] have presented a new method for generating distributions with an application to the exponential distribution. There is lots of life-time models which are obtained by compounding with Zero truncated Poisson distribution some of them are as follows

Kus (2007) [15] has presented exponential Poisson (EP) distribution having two-parameter by with zero truncated Poisson distribution with a decreasing failure rate compounded to exponential distribution. The Cumulative Distribution Function of PE distribution can be expressed as follows,

$$F(t;\beta,\lambda) = \frac{1}{\left(1-e^{-\lambda}\right)}\left[1-e^{\left\{-\lambda\left(1-e^{-\beta t}\right)\right\}}\right] \quad ; t>0, (\beta,\lambda)>0$$

(1.1)

While Barreto-Souza and Cribari-Neto (2009) [5] have proposed generalized EP distribution having the decreasing or increasing or upside-down bathtub shaped failure rate. This has generalization of distribution proposed by Kus (2007) [15] adding a power parameter to this distribution. Following a similar method, Percontini *et al*. (2013) [26] have introduced the five-parameter beta Weibull Poisson distribution, which is obtained the Weibull Poisson and beta distributions compounded together. Following the same trend, Cancho (2011) [6] has developed a novel distribution family also centered on the exponential distribution with an increasing failure rate function known as Poisson exponential (PE) distribution. The CDF of PE distribution is,

$$F\left(y;\lambda,\theta\right)=1-\frac{1-e^{\left\{-\theta\left(1-e^{-\lambda y}\right)\right\}}}{\left(1-e^{-\lambda}\right)} \quad ; y>0,\left(\lambda,\theta\right)>0$$

(1.2)

A Poisson-exponential having two-parameter and increasing failure rate has been defined by (Louzada-Neto *et al*., 2011) [6] by using the same approach as used by (Cancho, 2011) [18] under the Bayesian approach. Alkarni and Oraby (2012) have introduced a new lifetime family of distribution by compounding truncated Poisson distribution and a lifetime model with a decreasing failure rate. The cumulative distribution function of the Poisson generating family is

$$F_P\left(x;\lambda,\square\right)=1-\frac{1-\exp\left(-\lambda G\left(y,\square\right)\right)}{\left(1-e^{-\lambda}\right)} \quad ; \lambda>0$$

(1.3)

And PDF of Poisson-G family is

$$f_P\left(x;\lambda,\square\right)=\frac{1}{\left(1-e^{-\lambda}\right)}\lambda g\left(y,\square\right)\exp\left(-\lambda G\left(y,\square\right)\right); \lambda>0$$

(1.4)

Where $\square$ the parameter is space and $G(y,\square)$ is the CDF and $g(y,\square)$ PDF of any distribution. Using a parallel approach the Weibull power series class of distributions with Poisson has presented by (Morais & Barreto-Souza, 2011) [21]. Poisson exponential power (PEP) and Poisson inverse Weibull (PIW) both are introduced by (Joshi & Kumar, 2020) [11]. Mahmoudi and Sepahdar (2013) [23] have defined a new four-parameter distribution having decreasing, increasing, unimodal failure and bathtub-shaped rate known as exponentiated Weibull–Poisson (EWP) distribution which has been acquired by Poisson distributions and exponentiated Weibull (EW) compounded together. Similarly, Lu and Shi (2012) [19] have created the new compounding distribution which is known as the Weibull–Poisson distribution having variety of failure rate function. Joshi & Kumar (2020) [12] has developed Lindley exponential power distribution having variety of shape of failure rate function. Joshi & Kumar (2020) [13] has introduced the Half Logistic NHE, using the similar approach we have defined Poisson NHE distribution.

With this article, we aim to introduce a new distribution inserting an extra parameter to distribution given by (Nadarajah & Haghighi, 2011) [24] i.e. NHE distribution and achieve more flexibility and better fit to real data. Relevant properties and applications exhibited by the proposed distribution have been described to provide a better understanding to the distribution. The article shows following structure. In Section 2 we present the Poisson NHE distribution and its various mathematical and statistical properties. To estimate the model parameters, we have used some commonly used methods including Maximum likelihood estimation (MLE) constructing asymptotic confidence intervals from observed information matrix in section 3. In Section 4, a real data set has been analyzed to explore the applications and suitability of the proposed distribution. Here the distribution has been compared to existing distribution. Then in Section 5 conclusion has been presented.

The Poisson NHE (PNHE) distribution
We have presented an exponential distribution's extension (Nadarajah & Haghighi, 2011) [24] we have called it as Nadarajah and Haghighi (NHE) distribution. The CDF of NHE is defined as

$$F(s;\rho,\sigma)=1-e^{\{1-(1+\rho s)^{\sigma}\}}; \quad \rho,\sigma>0, \; s>0$$

(2.1)

The corresponding PDF is

$$f(s;\rho,\sigma)=\rho\sigma(1+\rho s)^{\sigma-1}e^{\{1-(1+\rho s)^{\sigma}\}}; \quad \rho,\sigma>0, \; s>0$$

(2.2)

Substituting (2.1) and (2.2) in (1.3) and (1.4) we get the CDF of Poisson NHE (PNHE) distribution, which is defined as

$$F\left(x;\alpha,\beta,\lambda\right)=1-\frac{1-\exp\left(-\lambda\exp\left\{\{1-(1+\alpha x)^{\beta}\}\right\}\right)}{\left(1-e^{-\lambda}\right)} \quad ;\alpha,\beta,\lambda>0,\ \lambda>0$$

(2.3)

And the PDF of Poisson NHE is

$$f(x;\alpha,\beta,\lambda)=\frac{\alpha\beta\lambda}{\left(1-e^{-\lambda}\right)}(1+\alpha x)^{\beta-1}e^{\{1-(1+\alpha x)^{\beta}\}}\exp\left(-\lambda\exp\left\{\{1-(1+\alpha x)^{\beta}\}\right\}\right);\ x>0$$

(2.4)

The PNHE's survival function is

$$S\left(x\right)=\frac{1-\exp\left(-\lambda\exp\left\{\{1-(1+\alpha x)^{\beta}\}\right\}\right)}{\left(1-e^{-\lambda}\right)} \quad ;\alpha,\beta,\lambda>0,\ x>0$$

(2.5)

A. Hazard function of PNHE distribution
The hazard rate function can be expressed as

$$h\left(x\right)=\frac{(1+\alpha x)^{\beta-1}e^{\{1-(1+\alpha x)^{\beta}\}}\exp\left(-\lambda\exp\left\{\{1-(1+\alpha x)^{\beta}\}\right\}\right)}{1-\exp\left(-\lambda\exp\left\{\{1-(1+\alpha x)^{\beta}\}\right\}\right)};\alpha,\beta,\lambda>0,\ x>0$$

(2.6)

B. Quantile function
Let Z be a positive random variable having a distribution function $F_z(z)$. Let $p\in(0,1)$, the pth-quantile of Z, denoted is

$$Q_Z(p)=F_Z^{-1}\left(p\right)$$
$$Q_Z(p)=\frac{1}{\alpha}\left[\left\{1-\ln\left(1-W\right)\right\}^{1/\beta}-1\right];\ 0<p<1$$
$$where\ W=1+\frac{1}{\lambda}\ln\left[1-\left(1-e^{-\lambda}\right)(1-p)\right]$$

(2.7)

The random deviate generation for the PNHE is,

$$x=\frac{1}{\alpha}\left[\left\{1-\ln\left(1-c\right)\right\}^{1/\beta}-1\right];\ 0<u<1$$
$$where\ c=1+\frac{1}{\lambda}\ln\left[1-\left(1-e^{-\lambda}\right)(1-u)\right]$$

(2.8)

C. Skewness and Kurtosis of PNHE distribution
The coefficient of skewness and kurtosis are important measures of dispersion in descriptive statistics. These measures are used mostly in data analysis for studying the shape of the distribution or data set. The Bowley's coefficient of skewness based on quartiles is,

$$S_k\left(Bowley\right)=\frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)},$$ and

Coefficient of kurtosis based on octiles given by (Moors, 1988) is

$$K_u\left(Moors\right)=\frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(3/4)-Q(1/4)}$$

Plots of probability density function and hazard rate function of PNHE (α,β,λ) with different values of parameter are presented in Figure 1.
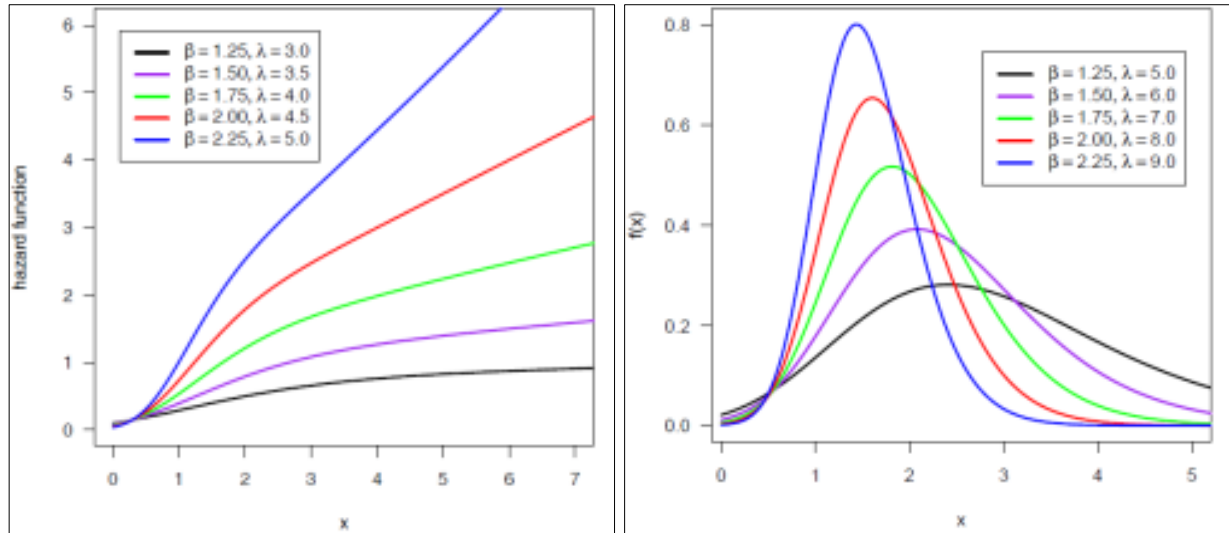

Fig 1: For fixed α and different values of β and λ, Plots of hazard function (left panel) and PDF (right panel).

Estimation methods of the model parameters
In this section, the parameters of the PNHE distribution are estimated

Maximum Likelihood Estimation Method
For the estimation of the parameter, the MLE method is the most commonly used method. Let, $x_1, x_2, ..., x_n$ is a random sample from $PNHE(\alpha, \beta, \lambda)$ and the likelihood function, $L(\alpha, \beta, \lambda)$ is given by,

$$L(\Psi; x_1, x_2...x_n) = f(x_1, x_2, ...x_n / \Psi) = \prod_{i=1}^{n} f(x_i / \Psi)$$

$$L(\alpha, \beta, \lambda) = \frac{\alpha\beta\lambda}{(1-e^{-\lambda})} \prod_{i=1}^{n} (1+\alpha x_i)^{\beta-1} e^{\{1-(1+\alpha x_i)^\beta\}} \exp\left(-\lambda \exp\left\{\{1-(1+\alpha x_i)^\beta\}\right\}\right); \quad x > 0$$

Now log-likelihood density is

$$\ell(\alpha, \beta, \lambda \mid \underline{x}) = n\ln(\alpha\beta\lambda) - n\ln\left(1-e^{-\lambda}\right) + (\beta-1)\sum_{i=1}^{n} \ln\left(1+\alpha x_i\right) + \sum_{i=1}^{n}\{1-(1+\alpha x_i)^\beta\}$$

$$- \lambda \sum_{i=1}^{n} \exp\left\{\{1-(1+\alpha x_i)^\beta\}\right\}$$

(3.1.1)

By differentiation of (3.1.1) with respect to α, β and λ we get,

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + (\beta-1)\sum_{i=1}^{n} \frac{x_i}{1+\alpha x_i} - \beta\sum_{i=1}^{n}(1+\alpha x_i)^{\beta-1} - \beta\lambda\sum_{i=1}^{n} x_i \exp\{1-(1+\alpha x_i)^\beta\}(1+\alpha x_i)^{\beta-1}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left(1+\alpha x_i\right) + \sum_{i=1}^{n}(1+\alpha x_i)^\beta \ln(1+\alpha x_i) + \lambda\sum_{i=1}^{n} \exp\{1-(1+\alpha x_i)^\beta\}(1+\alpha x_i)^\beta \ln(1+\alpha x_i)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \frac{ne^{-\lambda}}{1-e^{-\lambda}} - \sum_{i=1}^{n} \exp\{1-(1+\alpha x_i)^\beta\}$$

To obtain the MLEs $\hat{\alpha}, \hat{\beta} \text{ and } \hat{\lambda}$ of the respective parameters, α, β and λ, we equate the above equations to zero and solve for the parameters. The values for these parameters can be obtained with maximizing (3.1.1) with the help of platforms like R, Mathematica etc. Then calculation of observed information matrix has to be done for parameter's confidence interval estimation along with hypothesis testing. For the parameters α, β and λ, we can express the observed information matrix as,

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

Where

$$T_{11} = \frac{\partial^2 l}{\partial \alpha^2}, \; T_{12} = \frac{\partial^2 l}{\partial \alpha \partial \beta}, \; T_{13} = \frac{\partial^2 l}{\partial \alpha \partial \lambda}$$

$$T_{21} = \frac{\partial^2 l}{\partial \beta \partial \alpha}, \; T_{22} = \frac{\partial^2 l}{\partial \beta^2}, \; T_{23} = \frac{\partial^2 l}{\partial \beta \partial \lambda}$$

$$T_{31} = \frac{\partial^2 l}{\partial \lambda \partial \alpha}, \; T_{32} = \frac{\partial^2 l}{\partial \beta \partial \lambda}, \; T_{33} = \frac{\partial^2 l}{\partial \lambda^2}$$

Let $\Omega = (\alpha, \beta, \lambda)$ denote the parameter space, $T(\Omega)$ denotes Fisher's information matrix and the corresponding MLE of Ω as $\hat{\Omega} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$, then $(\hat{\Omega} - \Omega) \to N_3 \left[0, (T(\Omega))^{-1} \right]$. Maximizing the likelihood gives the observed information matrix via the Newton-Raphson algorithm and hence the variance-covariance matrix is obtained as,

$$\left[T(\Omega) \right]^{-1} = \begin{pmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha}, \hat{\beta}) & \text{cov}(\hat{\alpha}, \hat{\lambda}) \\ \text{cov}(\hat{\alpha}, \hat{\beta}) & \text{var}(\hat{\beta}) & \text{cov}(\hat{\beta}, \hat{\lambda}) \\ \text{cov}(\hat{\alpha}, \hat{\lambda}) & \text{cov}(\hat{\beta}, \hat{\lambda}) & \text{var}(\hat{\lambda}) \end{pmatrix}$$

(3.1.2)

Therefore for the parameters, the approximate 100(1-α) % confidence intervals can be built from the MLEs' asymptotic normality which is given below,

$$\hat{\beta} \pm Z_{\alpha/2} SE(\hat{\beta}), \; \hat{\alpha} \pm Z_{\alpha/2} SE(\hat{\alpha}) \text{ and, } \hat{\lambda} \pm Z_{\alpha/2} SE(\hat{\lambda})$$

And standard normal variate's upper percentile is denoted by $Z_{\alpha/2}$

Least-Square Estimation (LSE) Method
Here weighted least square estimators and ordinary least square estimators as given by Swain *et al*. (1988) [29] for estimating Beta distribution's parameters. With minimization of equation (3.2.1) with respect to unknown parameters α, β and λ, we can get least square estimators of parameters taken of the PHNE distribution which is given by,

$$B(X; \alpha, \beta, \lambda) = \sum_{i=1}^{n} \left[F(X_i) - \frac{i}{n+1} \right]^2$$

(3.2.1)

From a distribution function F(.) let $F(X_i)$ represent ordered random variables ($X_{(1)} < X_{(2)} < \dots < X_{(n)}$)'s distribution function and $\{X_1, X_2, \dots, X_n\}$ is a random sample (where n=size of sample). Minimization of equation (3.2.2) with respect to the unknown parameter α, β and λ, we can get respective least-square estimators $\hat{\alpha}, \hat{\beta}, \text{and} \hat{\lambda}$.

$$B(X; \alpha, \beta, \lambda) = \sum_{i=1}^{n} \left[1 - \frac{1 - \exp\left(-\lambda \exp\{1 - (1 + \alpha x_i)^\beta\}\right)}{\left(1 - e^{-\lambda}\right)} - \frac{i}{n+1} \right]^2 ; x \geq 0, (\alpha, \beta, \lambda) > 0.$$

(3.2.2)

With respect to α, β and λ, differentiation of (3.2.2) we get,

$$\frac{\partial B}{\partial \alpha} = \frac{\beta \lambda 2}{\left(1-e^{-\lambda}\right)} \sum_{i=1}^{n} x_i \left[1 - \frac{1-\exp\left(-\lambda K\left(x_i\right)\right)}{\left(1-e^{-\lambda}\right)} - \frac{i}{n+1} \right] K\left(x_i\right)\exp\left(-\lambda\left(x_i\right)K\right)(1+x_i\alpha)^{\beta-1}$$

$$\frac{\partial B}{\partial \beta} = \frac{2\lambda}{\left(1-e^{-\lambda}\right)} \sum_{i=1}^{n} \left[1 - \frac{1-\exp\left(-\lambda K\left(x_i\right)\right)}{\left(1-e^{-\lambda}\right)} - \frac{i}{n+1} \right] K\left(x_i\right)\exp\left(-\lambda K\left(x_i\right)\right)(1+\alpha x_i)^{\beta} \ln(1+\alpha x_i)$$

$$\frac{\partial B}{\partial \lambda} = 2 \sum_{i=1}^{n} \left[1 - \frac{1-\exp\left(-\lambda K\left(x_i\right)\right)}{\left(1-e^{-\lambda}\right)} - \frac{i}{n+1} \right] \left[\frac{K\left(x_i\right)\exp\left(-\lambda K\left(x_i\right)\right)}{1-e^{-\lambda}} - \frac{e^{-\lambda}\left\{1-\exp\left(-\lambda K\left(x_i\right)\right)\right\}}{\left(1-e^{-\lambda}\right)^2} \right]$$

Where

$$K\left(x_i\right) = \exp\{1-(1+\alpha x_i)^{\beta}\}$$

With minimization of following equation with respect to α, β and λ, we can get weighted LSE

$$J\left(X;\alpha,\beta,\lambda\right) = \sum_{i=1}^{n} w_i \left[F(X_{(i)}) - \frac{i}{n+1} \right]^2$$

$$w_i = \frac{1}{Var(X_{(i)})} = \frac{(2+n)(n+1)^2}{i(1+n-i)}$$

The weights w_i are

With minimization of equation (3.2.3) with respect to α, β and λ we can obtain weighted least square estimators of the following parameters

$$J\left(X;\alpha,\beta,\lambda\right) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[1 - \frac{1-\exp\left(-\lambda \exp\{1-(1+\alpha x_i)^{\beta}\}\right)}{\left(1-e^{-\lambda}\right)} - \frac{i}{n+1} \right]^2$$

(3.2.3)

3.3. Cramer-Von-Mises estimation (CVME) Method
With minimization of the following equation, the Cramer-Von-Mises estimators of α, β and λ are obtained.

$$A(X) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F\left(x_{i:n} \mid \alpha,\beta,\lambda\right) - \frac{2i-1}{2n} \right]^2$$

$$= \frac{1}{12n} + \sum_{i=1}^{n} \left[1 - \frac{1-\exp\left(-\lambda \exp\{1-(1+\alpha x_i)^{\beta}\}\right)}{\left(1-e^{-\lambda}\right)} - \frac{2i-1}{2n} \right]^2$$

(3.3.1)

With differentiation of (3.3.1) with respect to α, β and λ we obtain

$$\frac{\partial A}{\partial \alpha} = \frac{2\beta\lambda}{\left(1-e^{-\lambda}\right)} \sum_{i=1}^{n} x_i \left[1 - \frac{1-\exp\left(-\lambda K\left(x_i\right)\right)}{\left(1-e^{-\lambda}\right)} - \frac{2i-1}{2n} \right] K\left(x_i\right)\exp\left(-\lambda K\left(x_i\right)\right)(1+\alpha x_i)^{\beta-1}$$

$$\frac{\partial A}{\partial \beta} = \frac{2\lambda}{\left(1-e^{-\lambda}\right)} \sum_{i=1}^{n} \left[1 - \frac{1-\exp\left(-\lambda K(x_i)\right)}{\left(1-e^{-\lambda}\right)} - \frac{2i-1}{2n} \right] K(x_i)\exp\left(-\lambda K(x_i)\right)(1+\alpha x_i)^{\beta} \ln(1+\alpha x_i)$$

$$\frac{\partial A}{\partial \lambda} = 2 \sum_{i=1}^{n} \left[1 - \frac{1-\exp\left(-\lambda K(x_i)\right)}{\left(1-e^{-\lambda}\right)} - \frac{2i-1}{2n} \right] \left[\frac{K(x_i)\exp\left(-\lambda K(x_i)\right)}{1-e^{-\lambda}} - \frac{e^{-\lambda}\left\{1-\exp\left(-\lambda K(x_i)\right)\right\}}{\left(1-e^{-\lambda}\right)^2} \right]$$ Where

$$K(x_i) = \exp\{1-(1+\alpha x_i)^{\beta}\}$$

By solving $\frac{\partial A}{\partial \alpha} = 0, \frac{\partial A}{\partial \beta} = 0 \ and \ \frac{\partial A}{\partial \lambda} = 0$ simultaneously we get CVM estimators.

Application with a real dataset
For illustration of applicability of PNHE model we use an actual dataset. The data set is originally considered by (Bader & Priest, 1982). The data given represent the strength measured in GPA for single carbon fibers of 10mm in gauge lengths with sample size 63 and they are as follows:
1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020
The maximum likelihood estimates are calculated directly by using optim() function in R software (R Core Team, 2020) [31] and (Schmuller, 2017) [28] with maximization of likelihood function (3.1). In Table 1 we have demonstrated the MLE's and standard errors (SE) and 95% confidence interval for α, β and λ.

Table 1: MLE, SE and 95% confidence interval

Parameter	MLE	SE	95% ACI
alpha	0.5038	0.1871	(0.1371, 0.8705)
beta	1.8272	0.4369	(0.9709, 2.6835)
lambda	53.4573	5.9523	(41.7908, 65.1238)

In Figure 2 we have plotted P-P plot and Q-Q plot and it is seen that the proposed distribution fits the data very well.
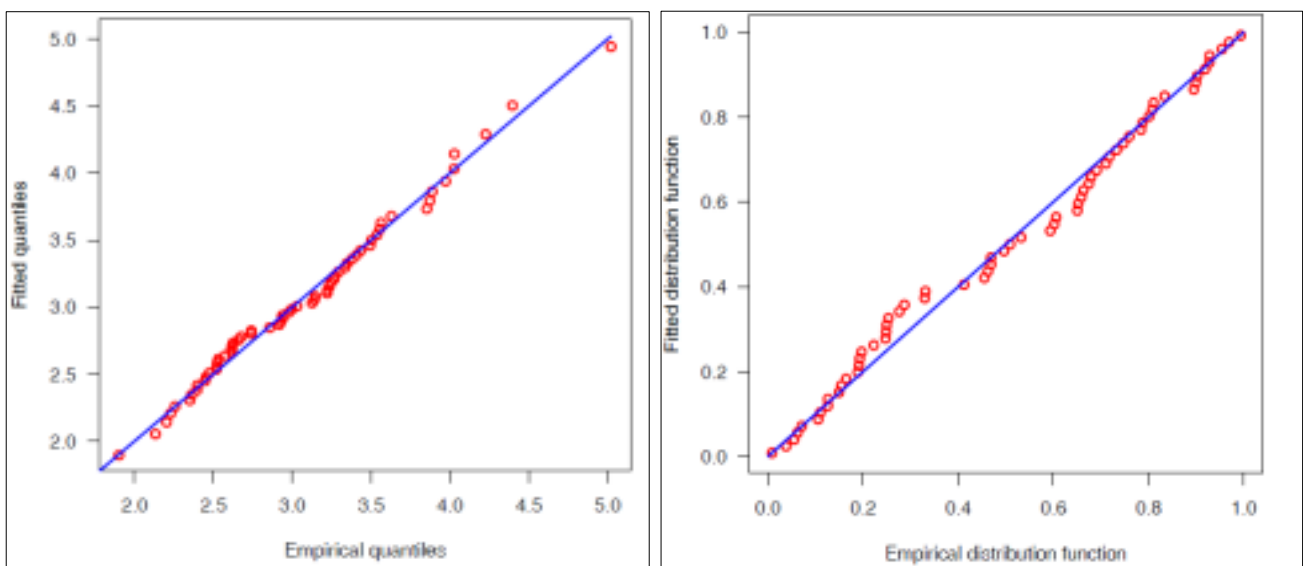

Fig 2: Plots of Q-Q (left side) and P-P (right side) of the PNHE distribution

The plots of profile log-likelihood function (Kumar & Ligges, 2011) for the parameters α, β and λ have been displayed in Figure 2 it is noticed that the ML estimates can be uniquely determined.
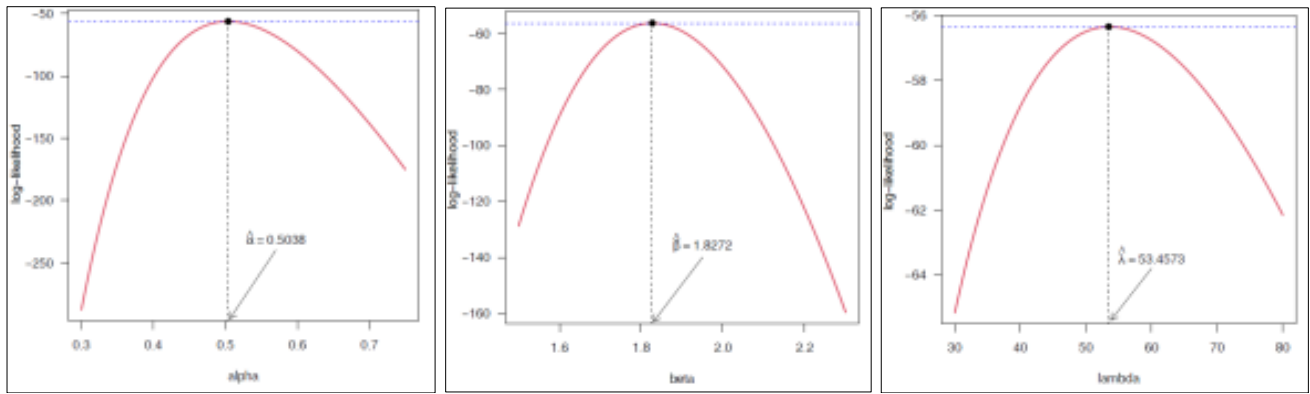
Fig 3: Plot of Profile log-likelihood function for the parameters α, β and λ

In Table 2 we have displayed the estimated value of the parameters of Poisson NHE distribution using MLE, LSE and CVE method and their corresponding negative log-likelihood and AIC criterion.

Table 2: Estimated parameters, log-likelihood and AIC

Method of Estimation	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	-LL	AIC
MLE	0.5038	1.8272	53.4573	56.3506	118.7013
LSE	0.0395	12.4637	14.7374	58.0094	122.0189
CVE	0.0342	14.3796	15.2396	57.8471	121.6942

In Table 3 we have presented value of Kolmogorov-Simnorov (KS), the Anderson-Darling (W) and the Cramer-Von Mises (A^2) statistics with their corresponding p-value of MLE, LSE and CVE method.

Table 3: The KS, W and A^2 statistics with a p-value

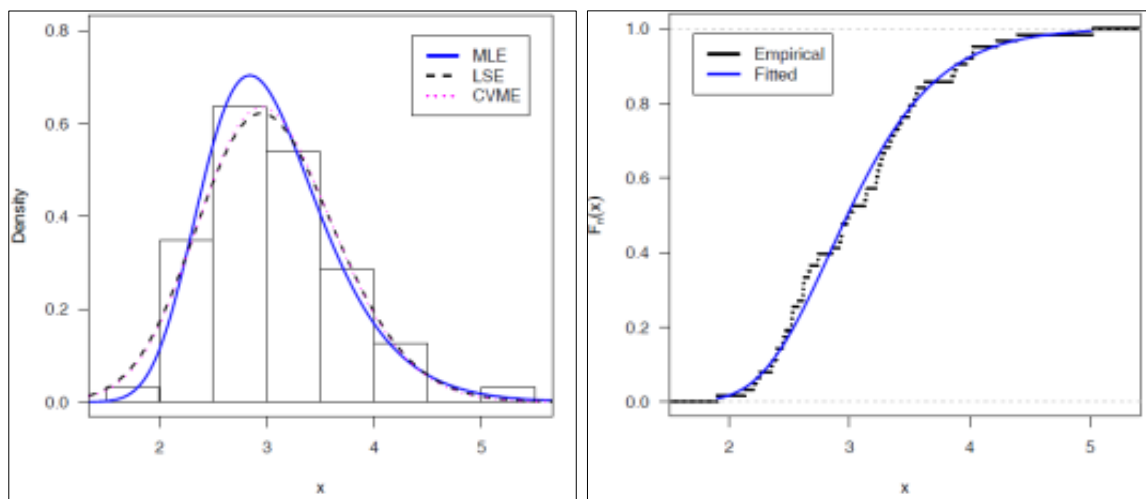| Method of Estimation	KS(p-value)	W(p-value)	A^2M(p-value)
MLE	0.0802(0.8120)	0.0632(0.7956)	0.3358(0.9087)
LSE	0.0667(0.9420)	0.0513(0.8701)	0.4241(0.8240)
CVE	0.0706(0.9124)	0.0504(0.8753)	0.4005(0.8478)

Fig 4: The Histogram and the PDF of fitted distributions of estimation methods MLE, LSE and CVM (left panel) and KS plot of PNHE distribution (right panel)

Illustration of the goodness of fit of the Poisson NHE distribution, we take these distribution for comparision,

1. Weibull Extension (WE) Model

The PDF of Weibull extension (WE) distribution introduced by (Tang *et al*., 2003) [30] with three parameters (α, β, λ) is

$$f_{WE}(x; \alpha, \beta, \lambda) = \lambda\beta \left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left(\frac{x}{\alpha}\right)^{\beta} \exp\left\{-\lambda\alpha\left(\exp\left(\frac{x}{\alpha}\right)^{\beta} - 1\right)\right\} \quad ; \quad x > 0$$

$\alpha > 0, \beta > 0$ and $\lambda > 0$

2. Exponential power (EP) distribution

The PDF of Exponential power (EP) distribution (Smith & Bain, 1975) is

$$f_{EP}(x) = \alpha\,\lambda^{\alpha}\,x^{\alpha-1}\,e^{(\lambda\,x)^{\alpha}}\,\exp\left\{1 - e^{(\lambda\,x)^{\alpha}}\right\} \quad ;(\alpha,\lambda) > 0, \quad x \geq 0$$

Here α and λ are the shape and scale parameters, respectively.

3. Poisson–exponential distribution (PE)

The probability density function of Poisson–exponential distribution was defined by (Louzada-Neto *et al*., 2011) [6] is

$$f(x) = \frac{\beta\lambda}{\left(1 - e^{-\lambda}\right)}\,e^{-\beta x}\,\exp\left(-\lambda e^{-\beta x}\right); \quad \beta > 0, \lambda > 0, x > 0$$

4. Chen distribution

Chen (2000) [7] has introduced Chain distribution having PDF as

$$f(x;\lambda,\theta) = \lambda\beta x^{\theta-1}e^{x^{\theta}}\,exp\left\{\lambda\left(1 - e^{x^{\theta}}\right)\right\} \quad ;(\lambda,\theta) > 0, x > 0$$

For the assessment of potentiality of the PNHE distribution, we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) which are presented in Table 4.

Table 4: Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
PNHE	56.3506	118.7013	125.1307	119.1080	121.2300
PE	57.2052	118.4105	122.6967	118.6105	120.0963
WE	61.9865	129.9731	136.4025	130.3798	132.5018
EP	69.3299	142.6598	146.9461	142.8533	144.3456
Chen	70.0133	144.0265	148.3128	144.2265	145.7124

Figure 5 shows the comparision between different distributions.
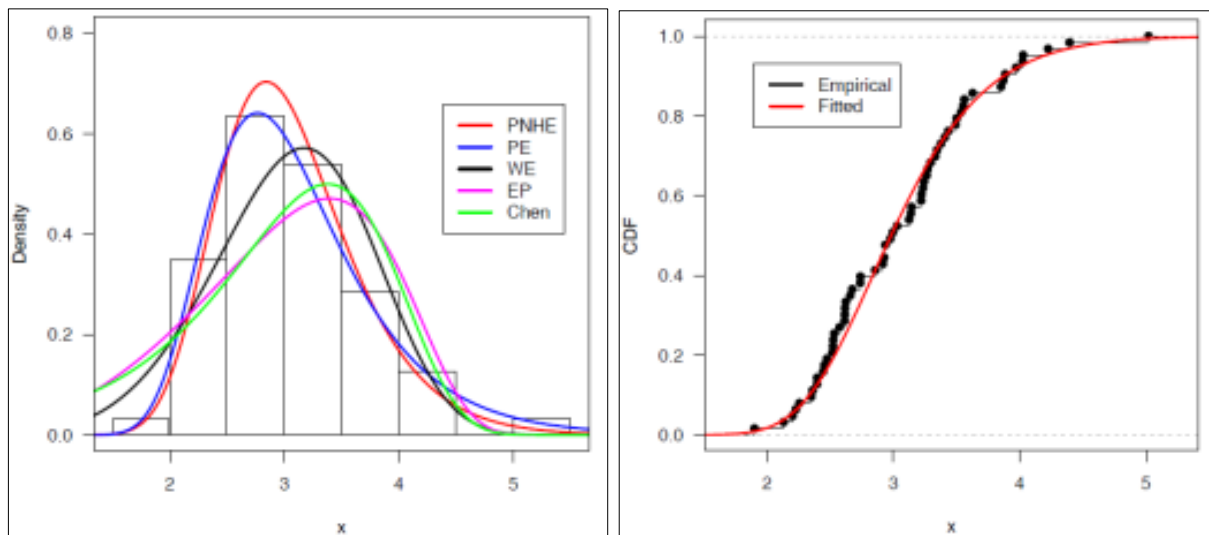

Fig 5: The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel)

In Table 5, to compare the goodness-of-fit of the PNHE distribution among various distribution we have shown the value of KS, W and A^2 statistics. It is observed that the distribution of PNHE has higher *p*-value and test statistic showing minimum value thus we can derive the conclusion that the PNHE distribution shows more consistency with results with higher reliability also showing better fit for the data from others taken for comparison.

Table 5: The goodness-of-fit statistics and their corresponding p-value

Model	KS(p-value)	W(p-value)	A²M(p-value)
PNHE	0.0620(0.9665)	0.0331(0.9661)	0.1988(0.9908)
PE	0.0971(0.5991)	0.0858(0.6606)	0.4686(0.7783)
WE	0.1042(0.5099)	0.1173(0.5077)	0.7373(0.5279)
EP	0.1365(0.2021)	0.2398(0.2021)	1.3735(0.2098)
Chen	0.1306(0.2464)	0.216(0.2387)	1.3143(0.2277)

Conclusions

In this work, we put forward the continuous Poisson NHE distribution. We have provided the distributional and mathematical properties of the proposed model. The curve of the PDF of the PNHE model can have uni-modal and positively skewed and the hazard function can exhibit increasing, constant and a broad variety of monotone failure rates. The P-P and Q-Q plots showed that the purposed distribution is quite better for fitting the real dataset. Using a real data set we have employed some well-known estimation methods including MLE and found that ML estimates are good than that of LSE and CVM. Also we have constructed the asymptotic confidence interval for MLEs. The application illustrates that the Poisson NHE distribution gives a consistently better fit and more flexible than those models taken for comparisons. It is hoped that this model will be alternative one to the field of probability and applied statistics.

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