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Lie symmetries for homogeneous and linear ordinary differential equation of first order

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Abstract
In applied Mathematics, differential equations are used to describe processes where dependence of a variable on other variables is not explicitly given rather the variable is given to be related to other variables in terms of rate of change. In such cases we need to integrate given equation so as to find function which describes the relationship between dependent and independent variables. There are various methods of finding this relationship. Lie symmetry method is based on finding transformations which help to reduce the complexity of given equation and transform the same to equation with lesser complexity. The transformed equation is then solved with usual methods and the solution is again transformed back to general solution of original differential equation. In this paper, lie transformations are obtained for linear and homogeneous differential equations of first order.

Keywords: Similarity transformations, lie transformations, lie group, linear differential equations, Homogenreous differential equation

1. Introduction
Lie symmetry solutions are devised by Sophus Lie in 19th century. This method is based on finding transformations to reduce the complexity of given differential equation. These transformations are called Lie transformations or point transformations. Lie transformations form a group called Lie group of transformations. Point transformations are then used to transform given equation to a much simpler form which can be integrated easily using known methods. In this work, the author has described the method of finding transformations for linear and homogeneous differential equations of first order. The method is then explained with the help of examples.

2. Related Work
Olver and Rosenau (1987) [12] introduced the concept of a weak symmetry group of a system of partial differential equations, that generalizes the "non classical" method introduced by Bluman and Cole (1969) for finding group-invariant solutions to partial differential equations. Bluman (1990) [1] found invariant solutions of an ordinary differential equation by solving an algebraic equation which derived from the given ODE and the infinitesimals of an admitted Lie group of transformations. Mahomad and Leach (1990) [9] showed that an nth (n>=3) order linear differential equation has exactly one of the (n + 1), (n + 2) or (n + 4) (the maximum) point symmetries. Clarkson and Olver (1994) analysed Chazy equation. Gazizov and Ibragimov (1998) classified two dimensional Jacobs-Jones model equations according to their symmetry groups. Burde (2002) used expanded transformations to the concept of similarity reductions of partial differential equations. Manno *et al*. (2006) determined necessary and sufficient conditions for a differential equation to be Lie Remarkable. Sophocleous and Leach (2010) [14] studied the partial differential equations which arise in the modelling of various financial instruments, risks, prices of commodities, etc. A remarkable number of these have rich algebraic structures. They illustrated the algebraic resolution of these problems with a number of examples. Carinena *et al*. (2014) [4] analyzed a class of Lie Systems on Dirac Manifolds. These manifolds are called Dirac-Lie system and these are associated with Dirac-Lie Hamiltonians. Martinot (2014) [8] explored some applications of the symmetries inherent to ordinary differential equations. He explained methods involving an integrating factor, homogeneous coordinates, or a reduction of order of a differential equation, in terms of symmetry transformations which are part of the theory of Lie groups.

Rehab and Shiekh (2015) [5] have investigated variable coefficients KdV equation. Paliathanasis and Leach (2016) [13] discussed the symmetry analysis and singularity analysis methods for the study of the integrability of nonlinear ordinary differential equations. Muhsen and Maan (2016) [10] applied Lie group analysis to second order Neutral Delay Differential Equations (NDDE). Hasan and Abd (2016) [6] used a coordinate transformation which takes a system of ordinary differential equations with no obvious solution to system of integrables.

3. Lie Symmetry Method

Let us consider the general form of first order ordinary differential equation:

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

Where x and y are independent and dependent variables respectively.

To solve (1), let us consider one parameter family of Lie group

$$\bar{x} = f_1(x, y; \varepsilon),\ \bar{y} = f_2(x, y; \varepsilon). \tag{2}$$

If equation (1) is invariant under transformation (2), then –

$$\frac{d\bar{y}}{d\bar{x}} = f(\bar{x}, \bar{y}), \tag{3}$$

Which gives

$$\frac{f_{2x} + f_{2y} f(x, y)}{f_{1x} + f_{1y} f(x, y)} = f\left(f_1(x, y, \varepsilon), f_2(x, y, \varepsilon)\right). \tag{4}$$

To solve (4), Lie considered extended form of Lie group called infinitesimal transformations. The infinitesimals are given by –

$$\xi = \left(\frac{df_1}{d\lambda}\right)_{\lambda=0} \tag{5}$$

$$\eta = \left(\frac{df_2}{d\lambda}\right)_{\lambda=0} \tag{6}$$

Using these infinitesimals, the given equation is transformed to canonical coordinates (r, s) by –

$$r_x \xi + r_y \eta = 0 \tag{7}$$

$$s_x \xi + s_y \eta = 1 \tag{8}$$

The equations (7) & (8) are then solved by method of characteristics to find r and s.

4. Example

In this section, Lie symmetries of a linear and homogeneous differential equation of first order are obtained.

4.1 Homogeneous Differential Equation

Consider first order homogeneous differential equation –

$$\frac{dy}{dx} = \frac{y^3 + x^3}{2x^2 y} \tag{9}$$

Comparing (9) with (1), we have –

$$f(x, y) = \frac{y^3 + x^3}{2x^2 y} \tag{10}$$

Let us consider one parameter family of Lie group as –

$$(\bar{x}, \bar{y}) = (e^{\lambda} x, e^{\lambda} y) \tag{11}$$

Here λ is a parameter

From (1), we get –

$$\frac{d(e^{\lambda} y)}{d(e^{\lambda} x)} = \frac{(e^{3\lambda} y^3 + e^{3\lambda} x^3)}{(2e^{2\lambda} x^2 e^{\lambda} y)}$$

$$\frac{e^{\lambda} dy}{e^{\lambda} dx} = \frac{e^{2\lambda}(y^3 + x^3)}{(2e^{2\lambda} x^2 y)}$$

Which gives

$$\frac{dy}{dx} = \frac{(y^3 + x^3)}{(2x^2 y)} \tag{12}$$

Therefore (9) is invariant under the transformation (11). Thus we have –

$$f_1(x, y, \lambda) = e^{\lambda x} \tag{13}$$

$$f_2(x, y, \lambda) = e^{\lambda y} \tag{14}$$

Now infinitesimals are given by–

$$\xi = \left(\frac{df_1}{d\lambda}\right)_{\lambda=0} = x \tag{15}$$

$$\eta = \left(\frac{df_2}{d\lambda}\right)_{\lambda=0} = y \tag{16}$$

From (7) and (8), equations in canonical coordinates are –

$$r_x x + r_y y = 0 \tag{17}$$

$$s_x x + s_y y = 1 \tag{18}$$

Lagrange's auxiliary equations for (17) are –

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dr}{0} \tag{19}$$

We get –

$$r = c_1 \tag{20}$$

and $$\frac{y}{x} = c_2 \tag{21}$$

Where c_1 and c_2 are arbitrary constants.
Therefore –

$$r = \frac{y}{x} \tag{22}$$

Lagrange's auxiliary equations for (18) are –

$$\frac{dx}{x} = \frac{dy}{y} = \frac{ds}{1} \tag{23}$$

Two independent solutions of (23) are –

$$s = \log c_3 x$$

and $$\frac{y}{x} = c_4 \tag{24}$$

Where c_3 and c_4 are arbitrary constants.
We take c_3=1 so as $s=logx$.
Therefore the Lie transformations for equation (9) are (y/x, $logx$)

4.2 Linear Differential Equation

First order linear differential equation is given by –

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{25}$$

Equation (25) is invariant under one parameter family of transformation –

$$(\bar{x}, \bar{y}) = (x, y + \lambda e^{-\int P(x)dx}) \tag{26}$$

Consider –

$$\frac{dy}{dx} + 2y = x^2 \tag{27}$$

(27) is a linear differential equation with y as dependent and x as independent variable.
Here –

$$P(x) = 2$$

Then –

$$\int P(x)dx = 2x$$

From (26), one parameter family of transformations is given by –

$$(\bar{x}, \bar{y}) = (x, y + \lambda e^{-2x})$$

Thus we have –

$$f_1(x, y, \lambda) = x \tag{28}$$

$$f_2(x, y, \lambda) = y + \lambda e^{-2x} \tag{29}$$

Now infinitesimals are given by –

$$\xi = \left(\frac{df_1}{d\lambda}\right)_{\lambda=0} = 0 \tag{30}$$

$$\eta = \left(\frac{df_2}{d\lambda}\right)_{\lambda=0} = e^{-2x} \tag{31}$$

From (7) and (8), equations in canonical coordinates are –

$$r_x.0 + r_y e^{-2x} = 0 \tag{32}$$

$$s_x.0 + s_y e^{-2x} = 1 \tag{33}$$

Solving (32) and (33), we get –
r = constant = 1, say

$$s = ye^{2x}$$

Therefore Lie transformations for (27) are (1, ye^{2x}).

5. Conclusion

Lie symmetry method is based on finding symmetries of a differential equation. These symmetries are transformations which help in reducing the complexity of a differential equation. Once the symmetries are obtained, the given equation is transformed to canonical form and then integrated to get the general solution. It is a systematic method and can be presented as an algorithm to solve differential equations of higher orders. In case of homogeneous and linear differential equations of first order, the symmetries are obtained in this work. Symmetries of Exact, Bernoulli and other first order ordinary differential equations can also be obtained with the same method.

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