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Study on differential equation models for solar heating

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Abstract
The aim in this paper is to fourth order finite difference method for a class of singular boundary value problems is presented. We discuss a direct method for solving singular boundary value problem. The finite difference methods are always a convenient choice for solving boundary value problems, because of their simplicity. The original differential equation is modified at the singular point.

Keywords: differential equation models, solar heating

1. Introduction

Singular boundary value problems in ordinary differential-difference equations occur in several models of non Newtonian fluids mechanics etc. In applied mathematics many problems lead to singular boundary value problems of the form

$$Ly \equiv y''(x) + \frac{k}{x}y'(x) + q(x)y(x) = r(x),\ 0 < x < 1,$$

$y'(0) = 0$ and $y(1) = \beta$,

which occur very frequently in the theory of thermal explosions and in the study of Electro-hydrodynamics. Such problems also arise in the study of generalized axially symmetric potentials after separation of variables has been employed. There is considerable interest on numerical methods on singular boundary value problems. Jamet [1] considered the usual three point finite difference scheme for singular boundary value problems and showed in the maximum norm that his scheme is $O(h^{1-\alpha})$ convergent. The usual classical three-point finite difference discretization for singular boundary value problems has been studied by Russell and Shampine [2-5].

Theoretical Formulations

We consider a singular two-point boundary value problem given by

$$Ly \equiv y''(x) + \frac{k}{x}y'(x) + q(x)y(x) = r(x), \qquad (1)$$

$$y'(0) = 0, \qquad (2)$$

$$y(1) = \beta. \qquad (3)$$

Jamet [1] has shown that for Eq. (1) the derivative boundary condition is imposed due to nature of physical situation of the problem. Due to the singularity at x = 0, we modify the problem near the singular point.

To set up difference equation of (1) divide [0, 1] into n equal parts, each of the length h, we have $x_i = ih$, $i = 0, 1, \ldots, n$. For simplicity, let $q(x_i) = q_i$, $r(x_i) = r_i$, $y(x_i) = yi$, $y'(x_i) = y'_i$ and $y''(x_i) = y''_i$.

Since x = 0 is singular point of Eq. (1), we first modify Eq. (1) at $x = x_0 = 0$ as follows:

$$y''(0) + \underset{x\to 0}{Lt}\frac{k}{x}h'(x) + q(0)y(0) = r(0)$$

Using L. Hospital rule, we have

$$\underset{x\to 0}{Lt}\frac{k}{x}y'(x) = ky''(0)$$

then we obtain

$(1+k)y''(x) + q(x)y(x) = r(x)$ at $x = 0$

Now, we describe a fourth order finite difference method, which leads to a tridiagonal system, which can be solved by Thomas Algorithm. By Taylor series expansion we obtain the CD formulas for y'_i , y_i'' assuming that y has continuous fourth order derivatives in the interval [0, 1]:

$$y_i'' \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi) \quad (4)$$

and

$$y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y'''(\eta) \quad (5)$$

where $\xi, \eta \in [x_{i-1}, x_{i+1}]$. Substituting (4) and (5) in (1) at $x = x_i$, we get the CD operator L_h, defined by

$$L_h y_i \equiv a_i y_{i+1} - b_i y_i + c_i y_{i-1} = d_i + \tau_i[y] \quad 1 \leqslant i \leqslant n-1, \quad (6)$$

where

$$a_i = \frac{1}{h^2} + \frac{k}{2hx_1}, \qquad b_i = \frac{2}{h^2} - q_i, \qquad c_i = \frac{1}{h^2} - \frac{k}{2hx_1}, \qquad d_i = r_i \quad (7)$$

and

$$\tau_i[y] = \frac{h^2}{12} h^{(4)}(\xi) + \frac{h^2 k}{6x_i} y'''(\eta),$$

where $\xi, \eta \in [x_{i-1}, x_{i-1}]$, here $\tau_i[y]$ are local truncation errors of the CD approximation. To obtain numerical solution of (1) by the CD operator L_h, we solve the system of equations formed by the three-term recurrence relation:

$$L_h y_i \equiv a_i y_{i+1} - b_i y_i + c_i y_{i-1} = d_i, \quad 1 \leqslant i \leqslant n-1. \quad (8)$$

By rewriting the CD formulas for y'_i, y_i'' in new form as given below:

$$y_i'' \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)} + R_i, \quad (9)$$

$$y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y''' + R_2, \quad (10)$$

where

$$R_1 = -\frac{2h^4 y^{(6)}(\xi)}{6!} \text{ and } R_2 = \frac{h^4 y^{(5)}(\eta)}{5!}$$

for $\xi, \eta \in [x_{i-1}, x_{i-1}]$. Substituting these y'_i , y_i'' from Eqs. (9) and (10) in (1) at $x = x_i$, we get the CD approximation in a form that includes all the $O(h^2)$ error terms:

$$L_h y_i - \frac{h^2}{12}\left(2\frac{k}{x_i} y_i''' + y_i^{(4)}\right) + \widetilde{R} = r_i, \quad (11)$$

L_h is the CD operator given as in (5) and $\widetilde{R} = R_1 + (k/x_i)R_2$.

By writing $q(x) = q$, $r(x) = r$, in (1) we obtain

$$y'' = r - qy - \frac{k}{x} y'.$$

Differentiating above equation with respect to x, we obtain

$$y''' = r' - \left[\frac{k}{x} y'' + \left(q - \frac{k}{x^2}\right) y' + q' y\right]. \quad (12)$$

Now differentiating (12) with respect to x, we get

$$y^{(4)} = r'' - \left[\frac{k}{x} y''' + \left(q - 2\frac{k}{x^2}\right) y'' + \left(2\frac{k}{x^3} + 2q'\right) y' + q'' y\right], \quad (13)$$

then

$$\begin{aligned} 2\frac{k}{x} y''' + y^{(4)} = &\left[2\frac{k}{x^2} - \frac{k^2}{x^2} - q\right] y'' + \left[-\frac{k}{x}\left(q - \frac{k}{x^2}\right) - 2\frac{k}{x^3} - 2q'\right] y' \\ &- \left[q'' + \frac{k}{x} q'\right] y + \frac{k}{x} r' + r''. \end{aligned} \quad (14)$$

Substituting (14) in (11), we get the equation

$$L_h y_i - \frac{h^2}{12}\left[\left(2\frac{k}{x_i^2} - \frac{k^2}{x_i^2} - q_i\right)y_i'' + \left(-\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) - 2\frac{k}{x_i^3} - 2q_i'\right)y_i' - \right.$$
$$\left. - \left(q_i'' + \frac{k}{x_i}q_i'\right)y_i\right] + \tilde{R} = r_i + \frac{h^2}{12}\left(\frac{k}{x_i}r_i' + r_i''\right). \tag{15}$$

We approximate the converted error terms in Eq. (15) by using for y_i'' and y_i' from Eqs. (4) and (5). Then adding these new approximations to $L_h y_i$ defined by (6) and (7), we obtain the fourth order operator

$$L_h^* y_i \equiv a_i^* y_{i+1} - b_i^* y_{i+1} + c_i^* y_{i-1} = d_i^* + \tau_i^*[y], \quad 1 \leqslant i \leqslant n-1. \tag{16}$$

where

$$a_i^* = a_i - \frac{1}{12}\left[2\frac{k}{x_i} - \frac{k^2}{x_i^2} - q_i\right] + \frac{h}{24}\left[\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) + 2\frac{k}{x_i^3} + 2q'\right],$$

$$b_i^* = b_i - \frac{1}{6}\left[2\frac{k}{x_i} - \frac{k^2}{x_i^2} - q_i\right] + \frac{h^2}{12}\left[q_i'' + \frac{k}{x_i}q_i'\right],$$

$$c_i^* = c_i - \frac{1}{12}\left[2\frac{k}{x_i} - \frac{k^2}{x_i^2} - q_i\right] - \frac{h}{24}\left[\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) + 2\frac{k}{x_i^3} + 2q'\right],$$

$$d_i^* = d_i - \frac{h^2}{12}\left[\frac{k}{x_i}r_i' + r_i''\right].$$

Here a_i, b_i, c_i, d_i are given in (7) and $\tau_i^*[y]$ are the local truncation errors of the Eq. (16), given by

$$\tau_i^*[y] = -\left[2\frac{k}{x_i^2} - \frac{k^2}{x_i^2} - q_i\right]\frac{h^4}{144}y^{(4)} + \left[\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) + 2\frac{k}{x_i^3} + 2q_i'\right]\frac{h^4}{72}y_i''' - \tilde{R},$$

where $\tilde{R} = R_i + (k/x_i)R_2 = O(h^4)$. We solve the system of equations formed by the three-term recurrence relationship:

$$L_h^* y_i \equiv a_i^* y_{i+1} - b_i^* y_i + c_i^* y_{i-1} = d_i^* \quad 1 \leqslant i \leqslant n-1. \tag{17}$$

2.3. Modification at singularity

The difference scheme (17) cannot be used at i = 0, as it is not defined at $x = x_0$. Hence we have modified Eq. (1) at singular point $x = x_0 = 0$ as in Eq. (1).

$$(1+k)y''(x) + q(x)y(x) = r(x), \qquad x = x_0 \tag{18}$$

Now we replace y''(x) with CD formulae (9) at $x = x_0$ (x = 0) in Eq. (18) and obtain

$$(1+k)\left(\frac{y_1 - 2y_0 + y_{-1}}{h^2} - \frac{h^2}{12}y_0^{(4)} + R_1\right) + q_0 y_0 = r_0. \tag{19}$$

Differentiating Eq. (2.18) twice with respect to x, we obtain

$$(1+k)y^{(4)} = r'' - 2q'y' - qy'' - q''y. \tag{20}$$

Substituting Eq. (20) in Eq. (19) and replacing y' and y'' with Eqs. (9) and (10) at x = 0, we obtain

$$\left(\frac{1+k}{h^2} + \frac{h}{12}q_0' + \frac{q_0}{12}\right)y_1 - \left(\frac{2(1+k)}{h^2} - \frac{h^2}{12}q_0'' - \frac{5}{6}q_0\right)y_0$$
$$+\left(\frac{1+k}{h^2} - \frac{h}{12}q_0' + \frac{q_0}{12}\right)y_{-1} = r_0 + \frac{h^2}{12}r_0'' + \tilde{R}, \tag{21}$$

where

$$\tilde{R} = -\frac{h^2}{12}q_0 R_1 - \frac{h^2}{6}q_0' R_2 - (1+k)R_1 + \frac{h^4}{144}q_0 y(\xi) + \frac{h^4}{72}2q_0' y'''(\eta)$$

which can be neglected.

To eliminate y_{-1} in Eq. (21) we use the boundary condition (2) y'(0) = 0 and applying finite difference approximation,

$$\frac{y_1 - y_{-1}}{2h} = 0.$$

Hence from Eq. (21)

$$y_0 = \frac{\frac{2(k+1)}{h^2} + \frac{q_0}{6}}{\frac{2(k+1)}{h^2} - \frac{5q_0}{6} - \frac{h^2}{12}q_0''} y_1 - \frac{r_0 + \frac{h^2}{12}r_0''}{\frac{2(k+1)}{h^2} - \frac{5q_0}{6} - \frac{h^2}{12}q_0''} \qquad (22)$$

Conclusion

We have described and demonstrated the applicability of the fourth order finite difference method by solving singular boundary value problems. First of all it is a direct method. Further it is simple, accurate, and easy to implement on computer.

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