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Study of Finite Permutation groups of finite simple groups

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Abstract
The main objective in this paper is to a discussion of several problems about primitive permutation groups which have been solved using the simple group classification. The classification of the finite simple groups has had far-reaching consequences for many branches of algebra.

Keywords: Finite Permutation Groups, Finite Simple

1. Introduction
The central problem of the theory of finite groups is to find for each positive integer n all groups (up to isomorphism) of order n. The solution to the more basic problem of classifying all finite simple groups was completed in 1980 and it is now the job of mathematicians in related fields to determine the consequences of this classification on their field of research. As an analogue of the preceding discussion of finite groups one might say that the central problem of the theory of finite permutation groups is to find for each positive integer n all permutation groups (up to equivalence) of degree n. (A permutation group of degree n is a subgroup of the symmetric group S_n of all permutations of the set n = {l, 2, . . . , n}.

Two permutation groups G, H of degree n are said to be equivalent if there is a group isomorphism $f : G \to H$ and a bijection $\phi : n \to n$ such that for all g in G and i in n, $(i^g)\phi = (i\phi)^{gf}$; that is G and H are equivalent if and only if they are isomorphic and act in the same way on n up to a relabelling of n.

Now let $G \leq S_n$: then G determines an equivalence relation on n by

$i \sim j \Leftrightarrow i^g = j$ for some g in G .

The equivalence classes are called orbits and G is called transitive if it has exactly one orbit.

Simple groups and primitive groups
The finite simple groups G may be conveniently listed as follows:

(a) $G = Z_p$, the cyclic group of order p, p a prime;

(b) $G = A_n$, the alternating group of degree n, that is the group of all even permutations of a set of size n, $n \geq 5$;

(c) G a group of Lie type: these may be divided into

(i) G = G (n, q), a classical group of dimension n "over" a field of q elements (these comprise linear, orthogonal, unitary and symplectic groups, six families in all),

(ii) G = G(q), an exceptional group over a field of q elements (10 families);

(d) G is one of the 26 sporadic simple groups.

Further information about groups of Lie type may be found in [1] for example, and about the sporadic simple groups in [2].

So far, the most useful result which has allowed the classification of the finite simple groups to be used to make headway with problems involving primitive permutation groups is a theorem of M. O'Nan and L.L. Scott.

It allows some questions about primitive permutation groups G to be reduced to the cases where G is a group of affine transformations of a vector space or $T \leq G \leq \text{Aut } T$ for some non-abelian simple group T. In essence their result is as follows (see [3], Theorem 1, for details and [4, 5] for details of the correction at part (ii)).

THEOREM 1. Let G be a primitive permutation group on a set X of n points, and let N be the socle of G (that is the group generated by all minimal normal subgroups of G). Then one of the following occurs.

(i) N is elementary abelian of order p^d and regular, $n = p^d$ where p is prime and $d \geq 1$, and $G \leq AGL(d, p)$ the group of affine transformations of N.

(ii) $N = T_1 \times \ldots \times T_m$ where T_i are isomorphic to a fixed nonabelian simple group T and $m \geq l$. Moreover if $m \geq 2$ then

(a) $n = |T|^{m-1}$ and the action of N on X is equivalent to its "diagonal action" on the cosets of a diagonal subgroup D = T of N, or

(b) $n = n^k{}_0$, $m = kr$, $G \leq G_0$. w S_k, where G_0 is primitive on Y of degree n_0 with, a minimal normal subgroup isoworphic to T^r, and the action of G on X is equivalent to its "product action" on Y^k. Further either T^r is the socle of G_0 [and G_0 satisfies (a) or r = 1), or r = 1 and T is regular on Y.

However using the classification of simple groups Cameron ([3], Theorem 1 (S)) showed that f(n) could be taken as $n^{c\log n}$ for some constant c, if one excludes $G \leq S_m$ wr S_k, in the product action, where S_m is acting on j-element subsets, $j \geq 1$, $k \leq 1$.

Table 1: Orders of primitive groups

		f(n)	(log f(n))*
Bochert wielandt Praeger, Saxl Babal	1989 1969 1981 1982	$n!/[(n+1)/2]!$ 4^n $\exp(4\sqrt{n}\log^2 n)$	n log n n $\sqrt{n}\log^2 n$
Cameron(1) Babai, Cameron Palfy(1)	1981 1982	$n^c\log^n$ n^c	$\log^2 n$ log n

(1)Bound for a subclass of primitive groups, see text.

He showed that even better results were possible if one was "careful". Finally (also using the simple group classification) Babai, Cameron and Palfy [6] showed that if the set of composition factors of the primitive group G of degree n contains no alternating group of degree greater than d and no classical group of dimension greater than d for some fixed positive integer d then |G| is polynomially bounded, that is $|G| \leq n^c$ for some constant c = c(d).

How many primitive permutation groups are there?

For each positive integer n the symmetric group S_n is primitive of degree n, as is the alternating group A_n for $n \neq 2$, while, for $n \leq h$, there are no other primitive groups of degree n. By l86l Mathieu [7] had shown that for each n = 5, ..., 33 there was a primitive (indeed a multiply transitive) group of degree n other than A_n and S_n.

Which permutations are excluded from primitive groups?

Let G be a primitive permutation group of degree n, $G \neq S_n$, A_n. We have seen that |G| is much smaller than n!. Are there many permutations $g \in S_n$ which are excluded from membership of any such G? In 2013 Jordan [8] showed that any permutation of prime order p with exactly one cycle of length p and at least 3 fixed points is excluded from membership of any primitive G ($\neq S_n$, A_n).

A permutation of $g \neq S_n$ is said to have type

$$t = 1^{a_1} 2^{a_2} \ldots n^{a_n}, \sum i a_i = n$$

if g has a_i cycles of length i for i = 1, . . . , n . Let T_n be the set of types t of permutations in S_n such that no permutation of type t may belong to any primitive permutation group G of degree n, $G \neq S_n$, A_n. Then Jordan's result is that $t = l^a p^1 \in T_n$ where p is prime and $a = n-p \geq 3$. Of course the previous section shows that T_n contains all types for "almost all" n since for almost all n no nontrivial primitive groups exist. Nevertheless the sets T_n are very much of interest as long as the classification of primitive permutation groups is incomplete. Jordan announced similar results, namely that $1^a p^q \in T_n$ where p is prime $p > q$, $1 \leq q \leq 5$ and $a > q+1$. Proofs of these results did not appear until W.A. Manning published them in 2019 and 2015. The best result of this period was obtained by Manning in 2018 (see [9] for a summary of results of the time), namely

$t = 1^a p^q \in T_n$, p prime, $5 < q \leq (p+l)/2$, $a > 4q - k$.

There were efforts to extend the range of q, but the lower bound on a had to increase markedly since the groups S_m and A_m, $m = q + (p+l)/2$, acting on unordered pairs contain an element of type $1^a p^q$ with $a = ((2q-p)^2-l)/8$.

In 2019, it was shown that these were the only groups containing such elements; namely if $2 < q < p$ and $a > (5q-4)/2$ then either $t = l^a p^q \in T_n$, or $n = m(m-l)/2$, where $= q + (p+l)/2$, $a = ((2q-p)^2-l)/8$, and in this case A_m and S_m on unordered pairs are the only non-trivial primitive groups containing elements of type $1^a p^q$.

Very recently using the simple group classification Liebeck and Saxl [10] have been able to classify all primitive permutation groups which contain elements of type $l^a p^q$ for $q < p$ and any $a \geq 0$.

Primitive groups with given rank

Let $G \leq S_n$ be primitive and let $X_1 = \{l\}, X_2, \ldots, X_r$ be the orbits of the stabilizer G_1 of 1 in n. Then G is said to have rank r. Clearly $r \geq 2$ and if r = 2, G is called 2-tvansitive. One of the first major consequences of the classification of the simple groups was the classification of all finite 2-transitive groups. A proof of this important result is given in Cameron's paper [3], Theorem 1 (S).

Primitive groups with given subdegree

With the notation of the previous section, the integers $d_i = |X_i|$, $1 \leq i \leq r$, are called the subdegvees of G. Let d be one of the subdegrees. It was conjectured by Sims that the order of G_1 is bounded by a function of d. That G is primitive is crucial for the conjecture to be true since the usual imprimitive action of $G = S_{(d+1)}$ wr S_k, of degree $n = (d+l)k$ has a subdegree d and the stabilizer of a point has order $d!(d+1)!^{(k-1)}$ (k-l)!. This conjecture has been much studied in the past 15 years and many partial results have been obtained. In particular Thompson [11] showed that G_1 has a normal subgroup of prime power order whose index in G_1 is bounded by a function of d. It has been possible to use the classification together with Thompson's result to prove the conjecture true. Although no attempt was made to get the best possible function it was shown that $|G_1| \leq \exp(d^2 o(d))$, [12].

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