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Studies on the Relativistic diffusion equation from stochastic quantization

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Abstract
In this paper, we only formulate the general notions of such a stochastic quantization and show how it works on simple, the models of relativistic and nonrelativistic particles interacting with an electromagnetic field. The development of the secondary stochastic quantization and its applications to the models with infinite degrees of freedom are left for a future work.

Keywords: Relativistic diffusion, stochastic quantization

1. Introduction
There are many different approaches to stochastic quantization. In this paper we propose another procedure of stochastic quantization, which in some sense generalizes the operator approach to the Fokker-Planck equation used in [1-10]. This new method of quantization gives a stochastic mechanics, which is not equivalent to quantum mechanics both in the manner of Nelson's stochastic quantization [5], and the Parisi-Wu stochastic quantization in the fictitious time. Rather we interpret the stochastic quantization from the point of view of the deformation quantization [6], i.e., as a deformation of an associative algebra of observables (smooth functions over a symplectic manifold) with an imaginary deformation parameter as opposed to an ordinary quantum mechanics with a real deformation parameter (the Planck constant).
This formulation of stochastic quantization allows us to apply the developed methods of quantum mechanics to the stochastic mechanics almost without any changing.

Formulations
We formulate the rules of stochastic quantization and define the main concepts of such a stochastic mechanics. Let us given a classical system with the Hamilton function H(t, x, p), where x^i and p_j are canonically conjugated with respect to the Poisson bracket positions and momenta

$$\{x^i, p_j\} = \delta^i_j, = \overline{1, d}, \tag{1}$$

where d is a dimension of the configuration space. As in quantum mechanics we associate with such a system the Hilbert space of all the square-integrable functions depending on x with the standard inner product.

$$\langle\psi|\varphi\rangle = \int d^d x \psi^*(x)\varphi(x), \tag{2}$$

Henceforth unless otherwise stated we consider only real-valued functions in this space.
In the Hilbert space we define the operators $\hat{x}^i$ and $\hat{p}_j$ such that

$$[\hat{x}^i, \hat{p}_j] = h\delta^i_j, \quad \hat{x}^{i+} = \hat{x}^i, \quad \hat{p}^+_j = -\hat{p}_j, \tag{3}$$

.

where $\tilde{h}$ is a small positive number and the cross denotes the conjugation with respect to the inner product (2). Define the Hamiltonian $\hat{H}(t, \hat{x}, \hat{p})$ by the von Neumann corresponding rules.

$x^i \rightarrow \hat{x}_i$, $p_j \rightarrow \hat{p}_j$. (4)

The state of the stochastic system is characterized by two vectors $|\psi\rangle$ and $|O\rangle$ from the Hilbert space with the evolution

$$\tilde{h}\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle, \quad \tilde{h}\frac{d}{dt}\langle O| = -\langle O|\hat{H}, \tag{5}$$

and the normalization condition.

$$\langle O|\psi\rangle = 1. \tag{6}$$

Define an average of the physical observable T(t, x, p) by the matrix element

$$\langle\hat{T}\rangle \equiv \langle O|\hat{T}(t, \hat{x}, \hat{p})|\psi\rangle, \tag{7}$$

where the operator $\hat{T}(t, \hat{x}, \hat{p})$ is constructed from T(t, x, p) by the corresponding rules (4). Then the Heisenberg equations for averages are

$$h\frac{d}{dt}\langle\hat{T}\rangle = \langle\partial_t\hat{T} + [\hat{T}, \hat{H}]\rangle. \tag{8}$$

By definition the probability density function is

$$\rho(x) = \langle O|x\rangle\langle x|\psi\rangle, \tag{9}$$

where $|x\rangle$ are the eigenvectors for the position operators corresponding to the eigenvalue x. The transition probability from the position x at the time t to x′ at the time t′ looks like

$$G(t', x'; t, x) = \langle O(t')|x'\rangle\langle x'|\hat{U}_{t',t}|x\rangle\frac{1}{\langle O(t)|x\rangle}, \tag{10}$$

where $\hat{U}_{t',t}$ is the evolution operator obeying the equations

$$h\partial_{t'}\hat{U}_{t',t} = \hat{H}\hat{U}_{t',t}, \qquad \hat{U}_{t',t} = \hat{1}. \tag{11}$$

The transition probability (10) possesses the property of a Markov process

$$G(t', x'; t, x) = \int d^d y G(t', x'; \tau, y)G(\tau, y; t, x). \tag{12}$$

By the standard means [7] we can construct a path integral representation of the transition probability (10). To this end we introduce auxiliary vectors |ipi in the Hilbert space such that

$$\hat{p}_j|ip\rangle = ip_j|ip\rangle, \quad \langle ip'|ip\rangle = \delta^d(p-p'), \quad \int\frac{d^d p}{(2\pi h)^d}|ip\rangle\langle ip| = \hat{1}. \tag{13}$$

In the coordinate representation we have

$$\langle x|ip\rangle = \exp\left\{-\frac{i}{h}p_i x^i\right\}. \tag{14}$$

Then inserting the unity partition (13) into the transition probability (10) we arrive at

$$\langle O(t+dt)|x'\rangle\langle x'|\hat{U}_{t+dt,t}|x\rangle\frac{1}{\langle O(t)|x\rangle}=$$

$$\left\langle x'|\exp\left\{\frac{dt}{h}\left[\hat{H}(t,\hat{x},\hat{p}+h\nabla\ln O(t,\hat{x}))+h\partial_t\ln O(t,\hat{x})\right]\right\}\right\rangle|x\rangle= \quad (15)$$

$$\int\frac{d^d p(t)}{(2\pi h)^d}\exp\left\{-\frac{i}{h}[p_i(t)x^i(t)+i(\overline{H}(t,x(t+dt),ip(t))+h\partial_t\ln O(t,x(t)))]dt\right\},$$

where x(t) = x, x(t + dt) = x′, x (t) = (x(t + dt) − x(t))/dt, O(t, x) = hO(t)|x⟩ and

$$\overline{H}(t,x,ip)=\langle x|\hat{H}(t,\hat{p}+h\nabla\ln O(t,\hat{x}),\hat{x})|ip\rangle\langle ip|x\rangle \quad (16)$$

is a qp-symbol of the Hamiltonian $\hat{H}$ with the momentum $\hat{p}$ + h∇ln $\hat{O}$.

The functional integral representation of the transition probability is obtained by the repeatedly use of the property (12) and the formula (15):

$$G(t',x';t,x)=\int\prod_{\tau\in(t,t')}d^d x(\tau)\prod_{\tau\in[t,t')}\frac{d^d p(\tau)}{(2\pi h)^d}\times$$

$$\exp\left\{-\frac{i}{h}\int_t^{t'-d\tau}d\tau[p_i(\tau)x^i(\tau)+i(\overline{H}(\tau,x(\tau+d\tau),ip(\tau))+h\partial_\tau\ln O(\tau,x(t)))]\right\}. \quad (17)$$

The property (12) guarantees that the functional integral representation (17) does not depend on what slices the time interval [t, t′] is cut [8]. We formulate the above stochastic mechanics in terms of the density operator

$\hat{\rho}$ = | ψ⟩⟨O|. (18)

From (5) and (6) it follows that

$$h\frac{d}{dt}\hat{\rho}=[\hat{H},\hat{\rho}],\quad Sp\hat{\rho}=1. \quad (19)$$

The averages are calculated as in quantum mechanics

$$\langle\hat{T}\rangle=Sp(\hat{\rho}\hat{T}). \quad (20)$$

The probability density function ρ(t, x) is the average of the projector |x⟩⟨x| and obeys the evolution law

$$h\partial_t\rho(t,x)=\langle x|[\hat{H},\hat{\rho}]|x\rangle. \quad (21)$$

The Fokker-Planck equation. Notice that from the definition (18) the density operator is idempotent, i.e.,

$$\hat{\rho}^2=\hat{\rho}. \quad (22)$$

By analogy with quantum mechanics one can say that such a density operator describes a pure state. The transition probability (10) is

$$G(t',x';t,x)=Sp(\hat{\rho}(t',t)|x'\rangle\langle x'|),\qquad \hat{\rho}(t,t)=\frac{|x\rangle\langle O|}{\langle O|x\rangle}, \quad (23)$$

where $\hat{p}$ (t′, t) obeys the von Neumann equation (19).

The formulation of the stochastic mechanics in terms of the density operator reveals that from the mathematical point of view the positions x^i are not distinguished over the momenta p_j as it seems from (3). The above stochastic quantization can be considered as a formal deformation of the algebra of classical observables in the manner of deformation quantization [6]. For a linear symplectic space the Moyal product is

$$f(z)*g(z)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{h}{2}\right)^n\omega^{a_1b_1}\cdots\omega^{a_nb_n}\partial_{a_1\ldots b_n}f(z)\partial_{b_1\ldots b_n}g(z), \quad (24)$$

where z ≡ (x, p), a_n, b_n = $\overline{1,2d}$, f(z) and g(z) are the Weil symbols, and ω^{ab} is the inverse to the symplectic 2-form ω^{ab}. The trace formula for averages is given by

$$\langle \hat{T} \rangle = \mathrm{Sp}\,(\hat{\rho}\hat{T}) = \int \frac{d^d x d^d p}{(2\pi h)^d} \sqrt{\det \omega_{ab}}\, \rho(x, p) T(p, x), \qquad (25)$$

where ρ(x, p) and T(p, x) are qp- and pq-symbols of the corresponding operators. For in- stance, the qp-symbol of the density operator is

$$\rho(x, ip) = \langle x|\, \hat{\rho}\, |ip\rangle\langle ip|x\rangle. \qquad (26)$$

Thus all the general results regarding deformation quantization of symplectic [9] and Poisson [10] manifolds, quantization of systems with constraints etc. are valid in such a stochastic mechanics.

Conclusion

In this paper we can consider the proposed quantization scheme from the position of deformation quantization. Then we investigate in this paper what happens when the algebra of observables is deformed by an imaginary parameter contrary to quantum mechanics with the real Planck constant.

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