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**Sunil Kumar Gupta**
Department of Mathematics, Govt. R.M.D. Girl's P.G. College Ambikapur, Chhattisgarh, India

**Deepak Kumar Tiwari**
Department of Mathematics, Shri Sai Baba Aadarsh Mahavidyalaya, Ambikapur, Chhattisgarh, India

**Corresponding Author:**
**Sunil Kumar Gupta**
Department of Mathematics, Govt. R.M.D. Girl's P.G. College Ambikapur, Chhattisgarh, India

# New fixed point results of integral type compatible mappings in cone metric space

**Sunil Kumar Gupta and Deepak Kumar Tiwari**

**Abstract**
In this paper, we establish some new fixed point results for integral type compatible mappings in cone metric space. Our results extend, modify and improve various well known results in the literatures.

**Keywords:** fixed point, compatible mapping, cone metric space

## Introduction

In 1976, Jungck [7] proved a common fixed point theorem for commutative mappings, generalizing the famous Banach contraction principle. Sesa [17] introduced the notion of weakly commutative maps. Also, in 1986, Jungck [8] introduced the notion of compatible mapping in order to generalize the concept of weak commutativity. Again, Pant [12, 13] defined R-weakly commutating maps and established some common fixed point theorems, assuming the continuity of at least one of the mappings. In 1997, H. K. Pathak and M. S. Khan [14] compared various types of compatible maps and gave a new common fixed points result. Kannan [10, 11] proved the existence of fixed point for a mapping that can have a discontinuity in a domain; however maps involved in each case were continuous at the fixed point.

In 1998, Jungck and Rhoades [9] defined a pair of self mappings to be weakly compatible if the commutes at their coincidence points. Then, applying these concepts, several authors have obtained coincidence point results for various classes of mappings in a metric space

Branciari [3] obtained a fixed point result for a single mapping satisfying an analogue of Banach's contraction principle for an integral-type inequality. The authors [2, 4, 15, 16, 22] proved some fixed point theorems involving more general contractive conditions. Also in [18], Suzuki shows that Meir-Keeler contractions of integral type are still Meir-Keeler contractions.

On the other hand, Huang and Zhang [5] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. In 2008, Abbas and Jungck [1] proved some common fixed point theorems for weakly compatible mappings in the setting of cone metric space; also K. Jha [6] proved common fixed point theorems in a cone metric space. In 2013, S. k. Tiwari, R. P. Dubey [19] proved some fixed point theorems for generalized contractive mapping in cone metric space and with A. K. Dubey [20] they gave common fixed point results in cone metric spaces. Recently in (July 2017), S. K. Tiwari, and Kaushik Das [21] extended some common fixed point results for contractive mappings in cone metric spaces.

Recently in 2015, Balaji R Wadekar, *et al.* [23] proved integral type common fixed point theorem in cone metric spaces.

In this paper, we establish common fixed point theorems for weakly compatible maps satisfying a contractive inequality of integral type in cone metric space. Our results generalize and stands some well known results in the literatures.

## 2. Preliminaries

We recall some standard notations and definitions

**Definition 2.1:** Let X be a real Banach space and P be a subset of X. Then P is called cone if,

i. P is closed and nonempty subset of X and P≠{0};

ii. $a,b \in R^{+}, x,y \in P \Rightarrow ax+by \in P$;

iii. $P \cap (-P) = \{0\}$

Given a cone $P \subseteq X$. We define a partial ordering $\leq$ on X with respect to P by $x \leq y \Leftrightarrow y-x \in P$. We shall write $x << y$ if $y-x \in \operatorname{int} P$. If $\operatorname{int} P \neq \phi$, then cone P will be solid. The cone P is said to be normal if there is a number K>0 such that for all $x, y \in E$

$0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$.

The least positive number K satisfying the above is called the normal constant of P.

The authors showed that there is no normal cone with normal constant M < 1 and for each K > 1. There are cone with normal constant M > K.

The cone P is called regular if every increasing sequence which is bounded from the above is convergent, that is if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq x_3 \leq x_4 \leq .......... \leq y$ for some $y \in X$, then there is $x \in X$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$.

The cone P is regular iff every decreasing sequence which is bounded from below is convergent.

**Definition 2.2:** Let X is a non empty set. Let $d : X \times X \to E$ be a mapping satisfies

$d_1$ : $0 < d(x,y)$ And $d(x,y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$

$d_2$ : $d(x,y) = d(y,x)$ For all $x, y \in X$

$d_3$ : $d(x,y) \leq d(x,z) + d(z,x)$ For all $x, y, z \in X$

Then d is called cone metric on X and (X, d) is called cone metric space.

**Definition 2.3:** Let X be a non empty set and X is a real Banach space, d is a mapping from X into itself such that d satisfies following conditions:

i. $\int_0^{d(x,y)} \varphi(t)dt \geq 0, \forall x, y \in X$

ii. $\int_0^{d(x,y)} \varphi(t)dt = 0 \Leftrightarrow x = y$

iii. $\int_0^{d(x,y)} \varphi(t)dt = \int_0^{d(y,x)} \varphi(t)dt$

iv. $\int_0^{d(x,y)} \varphi(t)dt \leq \int_0^{d(x,z)} \varphi(t)dt + \int_0^{d(z,y)} \varphi(t)dt$

Then d is called a cone metric on X and (X, d) is called cone metric space.

**Definition 2.2:** Let A & S be two mappings of a cone metric space (X, d) then it is said to be compatible if $\lim_{n\to\infty}\int_0^{d(ASx_n, SAx_n)} \varphi(t)dt = 0$ whenever $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = t$ and $\lim_{n\to\infty} Sx_n = t$ for some $t \in X$ of cone metric space.

Let A and S be two self mappings of (X, d) then it is said to be weakly compatible, if they commute at coincidence point that is $Ax = Sx$ for $x \in X$. It is easy to see that compatible mapping commutes at their coincidence points. It is note that compatible maps are weakly compatible but converse need not be true.

## 3. Main Results

**Theorem 3.1:** Let (X, d) be a cone metric space and P be a normal cone with normal constant k. Suppose $A, B$, and $C$ be three mappings from X into itself satisfying the condition

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \lambda \int_0^{[d(Ax,Cx)+d(By,Cy)+d(Ax,Cy)+d(Cx,By)+d(Cx,Cy)]} \varphi(t)dt$$

For all $x, y \in X$ and $\lambda \in [0,1)$ if $A(x) \cup B(x) \subseteq C(x)$ and C(x) is a complete subspace of X. Then the maps $A, B$ and $C$ have a unique point of coincidence in X. Moreover, if $(A,C)$ and $(B,C)$ are weakly compatible pairs; then $A, B$ and $C$ have a common unique fixed point in X.

**Proof:** Suppose since $A(x) \cup B(x) \subseteq C(x)$, starting with $x_0$ we define a sequence $\{y_k\}$ such that

$y_{2k} = Ax_{2k} = Cx_{2k+1}$

and $y_{2k+1} = Bx_{2k+1} = Cx_{2k+2}$

Consider that,

$$\int_0^{d(y_{2k},y_{2k+1})} \varphi(t)dt = \int_0^{d(Ax_{2k},Bx_{2k+1})} \varphi(t)dt$$

$$\leq \lambda \int_0^{[d(Ax_{2k},Cx_{2k})+d(Bx_{2k+1},Cx_{2k+1})+d(Ax_{2k},Cx_{2k+1})+d(Cx_{2k},Bx_{2k+1})+d(Cx_{2k},Cx_{2k+1})]} \varphi(t)dt$$

$$\leq \lambda \int_0^{[d(y_{2k},y_{2k-1})+d(y_{2k+1},y_{2k})+d(y_{2k+1},y_{2k})+d(y_{2k-1},y_{2k+1})+d(y_{2k-1},y_{2k})]} \varphi(t)dt \Rightarrow \int_0^{d(y_{2k},y_{2k+1})} \varphi(t)dt \leq \frac{3\lambda}{1-2\lambda} \int_0^{d(y_{2k-1},y_{2k})} \varphi(t)dt$$

$$\therefore \int_0^{d(y_{2k},y_{2k+1})} \varphi(t)dt \leq h \int_0^{d(y_{2k-1},y_{2k})} \varphi(t)dt$$

Where $h = \frac{3\lambda}{1-2\lambda} < 1.$

Similarly we can show that

$\int_0^{d(y_{2k+2},y_{2k+1})} \varphi(t)dt \leq h' \int_0^{d(y_{2k+1},y_{2k})} \varphi(t)dt$ where $h' = \frac{3\lambda}{1-\lambda} < 1.$

Therefore, for all k,

$$\int_0^{d(y_{2k+2},y_{2k+1})} \varphi(t)dt \leq h \int_0^{d(y_{2k+1},y_{2k})} \varphi(t)dt$$

$$\leq h^2 \int_0^{d(y_{2k},y_{2k-1})} \varphi(t)dt$$

$\leq$ …………………..

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$$\leq h^{2k+1} \int_0^{d(y_0,y_1)} \varphi(t)dt.$$

Now for any m>k,

$$\int_0^{d(y_{2k},y_{2m})} \varphi(t)dt \leq \int_0^{[d(y_k,y_{2k+1})+d(y_{2k+1},y_{2k+2})+\dots\dots\dots+d(y_{2m-1},y_{2m})]} \varphi(t)dt$$

$$\leq h^{2k} \int_0^{d(y_0,y_1)} \varphi(t)dt + h^{2k+1} \int_0^{d(y_0,y_1)} \varphi(t)dt + \dots\dots\dots\dots + h^{2m-1} \int_0^{d(y_0,y_1)} \varphi(t)dt$$

$$= (h^{2k} + h^{2k+1} + h^{2k+2} \dots\dots\dots\dots) \int_0^{d(y_0,y_1)} \varphi(t)dt$$

$$= \frac{h^{2k}}{1-h} \int_0^{d(y_0,y_1)} \varphi(t)dt.$$

Since *P* is normal with normal constant *k*.

Therefore, $\left\| \int_0^{d(y_{2k},y_{2m})} \varphi(t)dt \right\| \leq \frac{h^{2k}}{1-h} k \left\| \int_0^{d(y_0,y_1)} \varphi(t)dt \right\| \to 0$ as $k \to \infty$

Then, $\int_0^{d(y_{2k},y_{2m})} \varphi(t)dt \to 0$ as $k, m \to \infty$ so by $\{y_{2k}\} = \{Cx_{2k-1}\}$ is a Cauchy sequence.

Since *C(X)* is a complete subspace of X, so $\exists v \in C(X)$ such that

$\lim_{k\to\infty} y_{2k} = v$ i.e. $Cx_{2k} \to v$ as $k \to \infty.$

Consequently, we can find *u* in *X* such that $C(u) = v.$

We shall show that $Cu = Au = Bu.$

Consider,

$$\int_0^{d(Au,y_{2k+1})} \varphi(t)dt = \int_0^{d(Au,Bx_{2k+1})} \varphi(t)dt$$

$$\leq \lambda \int_0^{[d(Au,Cu)+d(Bx_{2k+1},Cx_{2k+1})+d(Au,Cx_{2k+1})+d(Cu,Bx_{2k+1})+d(Cu,Cx_{2k+1})]} \varphi(t)dt$$

$$\leq \lambda \int_0^{[d(Au,Cu)+d(y_{2k+1},y_{2k})+d(Au,y_{2k})+d(Cu,y_{2k+1})+d(Cu,y_m)]} \varphi(t)dt.$$

Hence, we have

$$\left\|\int_0^{d(Au,Cu)}\varphi(t)dt\right\| \le k\lambda\left\|\int_0^{[d(Au,Cu)+d(y_{2k+1},y_{2k})+d(Au,y_{2k})+d(Cu,y_{2k+1})+d(Cu,y_m)]}\varphi(t)dt\right\|$$

If $k\to\infty$, then $\left\|\int_0^{d(Au,Cu)}\varphi(t)dt\right\| = 0$, thus $Au = Cu$.

Similarly we can show that $Bu = Cu$.

Therefore, $Au = Bu = Cu = v$.

Now we show that $A, B, C$ have a unique point of coincidence.

For this we assume that there exists another point $x^* \in X$ such that,

$Ax^* = Bx^* = Cx^* = y^*$

Then we have,

$$\int_0^{d(y^*,v)}\varphi(t)dt = \int_0^{d(Ax^*,Bu)}\varphi(t)dt$$

$$\le \lambda\int_0^{[d(Ax^*,Cx^*)+d(Bu,Cu)+d(Ax^*,Cu)+d(Bu,Cx^*)+d(Cx^*,Cu)]}\varphi(t)dt$$

$$= \lambda\int_0^{[d(y^*,y^*)+d(v,v)+d(y^*,v)+d(y^*,v)+d(y^*,v)]}\varphi(t)dt$$

$\le 3\lambda\int_0^{d(y^*,v)}\varphi(t)dt$ which gives a contraction.

Hence $\left\|\int_0^{d(y^*,v)}\varphi(t)\right\| = 0.$ so we have $y^* = v.$

Thus the point of coincidence is unique.

If pairs $(A,C)$ and $(B,C)$ are weakly compatible then

$Av = ACu = CAu = Cv$

And $Bv = BCu = CBu = Cv.$

Therefore, $Av = Bv = Cv = w$ (say). This shows that w is another point of coincidence. Therefore by uniqueness, we must have

$w = v$ i.e. $Av = Bv = Cv = v.$

Thus $v$ is a unique common fixed point of self maps $A, B$ and $C$ in $X$ .

This completes the proof of the theorem.

**Theorem 3.2:** Let $(X,d)$ be a cone metric space and $P$ be a normed cone with normal constant $K$. Suppose $A, B \,\&\, C$ be three mappings from $X$ into itself satisfy the condition

$$\int_0^{d(Ax,By)}\varphi(t)dt \le \alpha\int_0^{d(Ax,Cx)}\varphi(t)dt + \beta\int_0^{[d(Ax,Cx)+d(By,Cy)]}\varphi(t)dt + \gamma\int_0^{[d(Ax,Cy)+d(By,Cx)]}\varphi(t)dt \quad \ldots(3.2.1)$$

$\forall x, y \in X$ and for all $\alpha, \beta, \gamma \in [0,1)$ such that $\alpha + 2\beta + 2\gamma < 1$. If $A(X)\bigcup B(X) \subseteq C(X)$ and $C(X)$ is a complete subspace of $X$ . Then the maps $A, B \,\&\, C$ have a coincidence point in $X$ . Moreover, if $(A,C)$ and $(B,C)$ are weakly compatible pairs, then $A, B \,\&\, C$ have a unique common fixed point in $X$ .

**Proof:** Let $x_0$ be an arbitrary point in X and we define a sequence $\{y_k\}$ in X such that

$y_{2k} = Ax_{2k} = Cx_{2k+1} \qquad k = 0,1,2,3.........$

and $y_{2k+1} = Bx_{2k+1} = Cx_{2k+2} \qquad k = 0,1,2,3.........$

Consider that,

$=$

$$\int_0^{d(y_{2k},y_{2k+1})}\varphi(t)dt = \int_0^{d(Ax_{2k},Bx_{2k+1})}\varphi(t)dt$$

$$\le \alpha\int_0^{d(Cx_{2k},Cx_{2k+1})}\varphi(t)dt + \beta\int_0^{[d(Ax_{2k},Cx_{2k})+d(Bx_{2k+1},Cx_{2k+1})]}\varphi(t)dt + \gamma\int_0^{[d(Ax_{2k},Cx_{2k+1})+d(Bx_{2k+1},Cx_{2k})]}\varphi(t)dt$$

$$= \alpha\int_0^{d(Ax_{2k},Bx_{2k+1})}\varphi(t)dt + \beta\int_0^{[d(y_{2k},y_{2k-1})+d(Bx_{2k+1},Ax_{2k})]}\varphi(t)dt + \gamma\int_0^{[d(y_{2k},y_{2k+1})+d(y_{2k},y_{2k-1})]}\varphi(t)dt$$

$$= \alpha\int_0^{d(y_{2k},y_{2k-1})}\varphi(t)dt + \beta\int_0^{[d(y_{2k},y_{2k-1})+d(y_{2k},y_{2k-1})]}\varphi(t)dt + \gamma\int_0^{[d(y_{2k},y_{2k+1})+d(y_{2k},y_{2k-1})]}\varphi(t)dt$$

$$\Rightarrow \int_0^{d(y_{2k},y_{2k+1})} \varphi(t)dt \leq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} \int_0^{d(y_{2k},y_{2k-1})} \varphi(t)dt$$

$$\therefore \int_0^{d(y_{2k},y_{2k+1})} \varphi(t)dt \leq h \int_0^{d(y_{2k},y_{2k-1})} \varphi(t)dt$$

Where $h = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1.$

Similarly we can show that

$$\int_0^{d(y_{2k+2},y_{2k+1})} \varphi(t)dt \leq h' \int_0^{d(y_{2k+1},y_{2k})} \varphi(t)dt \text{ where } h' = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1.$$

Therefore, for all $\forall k \in N$ we have,

$$\int_0^{d(y_{2k+2},y_{2k+1})} \varphi(t)dt \leq h \int_0^{d(y_{2k},y_{2k+1})} \varphi(t)dt$$
$$\leq h^2 \int_0^{d(y_{2k},y_{2k+1})} \varphi(t)dt$$
$\leq$……………………..
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$$\leq h^{2k+1} \int_0^{d(y_0,y_1)} \varphi(t)dt$$

Now for any m>k,

$$\int_0^{d(y_{2k},y_{2m})} \varphi(t)dt \leq \int_0^{d(y_k,y_{2k+1})+d(y_{2k+1},y_{2k+2})+\dots\dots+d(y_{2m-1},y_{2m})} \varphi(t)dt$$
$$\leq h^{2k} \int_0^{d(y_0,y_1)} \varphi(t)dt + h^{2k+1} \int_0^{d(y_0,y_1)} \varphi(t)dt + \dots\dots\dots + h^{2m-1} \int_0^{d(y_0,y_1)} \varphi(t)dt$$
$$\leq (h^{2k} + h^{2k+1} + h^{2k+2} \dots\dots\dots) \int_0^{d(y_0,y_1)} \varphi(t)dt$$
$$= \frac{h^{2k}}{1-h} \int_0^{d(y_0,y_1)} \varphi(t)dt.$$

Since *P* is normal with normal constant *k*.

Therefore, $\left\| \int_0^{d(y_{2k},y_{2m})} \varphi(t)dt \right\| \leq \frac{h^{2k}}{1-h} k \left\| \int_0^{d(y_0,y_1)} \varphi(t)dt \right\|.$

Then, $\int_0^{d(y_{2k},y_{2m})} \varphi(t)dt \to 0$ as $k, m \to \infty$ with $0 < \alpha + 2\beta + 2\gamma < 1$ so $\{y_{2k}\} = \{Cx_{2k}\}$ is a Cauchy sequence.

Now, since *C(X)* is a complete subspace of *X*, so $\exists v \in C(X)$ such that

$\lim_{k\to\infty} y_{2k} = v$ i.e. $Cx_{2k} \to v$ as $k \to \infty.$

Consequently, we can find u in X such that $C(u) = v.$

We shall show that $Cu = Au = Bu.$

For this we consider,

$$\int_0^{d(Au,y_{2k+1})} \varphi(t)dt = \int_0^{d(Au,Bx_{2k+1})} \varphi(t)dt$$

$$\leq \alpha \int_0^{d(Cu,Cx_{2k+1})} \varphi(t)dt + \beta \int_0^{[d(Au,Cu)+d(Bx_{2k+1},Cx_{2k+1})]} \varphi(t)dt + \gamma \int_0^{[d(Au,Cx_{2k+1})+d(Bx_{2k+1},Cu)]} \varphi(t)dt$$

$$\leq \alpha \int_0^{d(Cu,Cx_{2k+1})} \varphi(t)dt + \beta \int_0^{[d(Cx_{2k+1},Cu)+d(Au,Cx_{2k+1})]} \varphi(t)dt + \gamma \int_0^{[d(Au,y_{2k})+d(y_{2k+1},Cu)]} \varphi(t)dt .$$

$$\leq \alpha\int_0^{d(Cu,Cx_{2k+1})}\varphi(t)dt+\beta\int_0^{[d(Cx_{2k+1},Cu)+d(Bx_{k+1},Au)]}\varphi(t)dt+\gamma\int_0^{[d(Au,y_{2k})+d(Cx_{2k+1},Cu)]}\varphi(t)dt$$

$$\Rightarrow \int_0^{d(Au,y_{2k+1})}\varphi(t)dt \leq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\int_0^{d(Cx_{2k+1},Cu)}\varphi(t)dt$$

Since $P$ is a normal cone with normal constant $k$, so

$$\left\|\int_0^{d(Au,y_{2k+1})}\varphi(t)dt\right\| = \left\|\int_0^{d(Au,Bx_{2k+1})}\varphi(t)dt\right\| \leq k.\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\left\|\int_0^{d(Cx_{2k+1},Cu)}\varphi(t)dt\right\| \to 0 \text{ as } k\to\infty$$

Therefore, for larger $n$ we get

$$\int_0^{d(Cu,Au)}\varphi(t)dt \leq \int_0^{d(v,Cx_{2k+2})}\varphi(t)dt \leq \int_0^{d(v,u)}\varphi(t)dt = 0$$

Which leads to $\int_0^{d(Cu,Au)}\varphi(t)dt = 0$ and hence $Cu = v = Au.$

Similarly we can show that

$Cu = v = Bu$ therefore we have,

$v = Cu = Au = Bu$ .

i.e. $u$ is a coincidence point of mappings $A, B, C$ .

Since $(A,C)$ and $(B,C)$ are weakly mapping at point $u$ and contactive condition (3.2.1), we have

$$\int_0^{d(AAu,Au)}\varphi(t)dt = \int_0^{d(AAu,Bu)}\varphi(t)dt$$

$$\leq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\int_0^{d(CAu,Cu)}\varphi(t)dt$$

$$\leq \int_0^{d(CAu,Cu)}\varphi(t)dt$$

$$= \int_0^{d(ACu,Au)}\varphi(t)dt$$

$$= \int_0^{d(AAu,Au)}\varphi(t)dt$$

i.e. $\int_0^{d(AAu,Au)}\varphi(t)dt \leq \int_0^{d(AAu,Au)}\varphi(t)dt$ a contradiction. Therefore

$AAu = Au$ .

Hence $Au = AAu = ACu = CAu.$

$\Rightarrow AAu = CAu = Au = v.$

Therefore $Au(=v)$ is a common fixed point of $A$ and $C$ .

Similarly, we can show that

$Bu = BBu = BCu = CBu$

Implies that $BBu = CBu = Bu = v.$

Therefore, $Bu = Au(=v)$ is a common fixed point of $B$ and $C$ . Hence by the above discussion we concluded that $A, B, C$ have a common fixed point $v$.

**Uniqueness**: Let $v^*$ be any other fixed point of $A, B, C$ consider that

$$\int_0^{d(v,v^*)}\varphi(t)dt = \int_0^{d(Av,Bv^*)}\varphi(t)dt$$

$$\leq \alpha\int_0^{d(Cv,Cv^*)}\varphi(t)dt+\beta\int_0^{[d(Av,Cv)+d(Bv^*,Cv^*)]}\varphi(t)dt+\gamma\int_0^{[d(Ax^*,Cu)+d(Bu,Cx^*)]}\varphi(t)dt$$

$$\leq \alpha\int_0^{d(Cv,Cv^*)}\varphi(t)dt+\beta\int_0^{[d(Cv^*,Cv)+d(Av,Cv^*)]}\varphi(t)dt+\gamma\int_0^{[d(y^*,v)+d(v,y^*)]}\varphi(t)dt$$

$$\leq \alpha\int_0^{d(Cv,Cv^*)}\varphi(t)dt+\beta\int_0^{[d(Cv^*,Cv)+d(Av,Cv^*)]}\varphi(t)dt+\gamma\int_0^{d(y^*,y^*)}\varphi(t)dt$$

$\leq (\alpha+\beta)\int_0^{d(Cv,Cv^*)}\varphi(t)dt+\beta\int_0^{d(Av,Cv^*)}\varphi(t)dt$ which gives a contradiction.

Hence $\left\|\int_0^{d(v,v^*)}\varphi(t)dt\right\| = 0 \Rightarrow v = v^*$ is a contradiction therefore $A, B$ and $C$ have a unique common fixed point in $X$ .

**Theorem 3.3:** Let $(X,d)$ be a cone metric space and $P$ be a normed cone with normal constant $K$. Suppose $A,B\&C$ be three maps from $X$ to itself satisfy the condition $\int_0^{d(Ax,By)}\varphi(t)dt \le \alpha\int_0^{[d(Ax,Cx)+d(By,Cy)]}\varphi(t)dt + \beta\int_0^{[d(Cx,Cy)+d(Ax,Cy)+d(By,Cx)]}\varphi(t)dt$ …(3.3.1)
Where $\alpha+2\beta<1$ is a constant for all $\alpha,\beta\in[0,1)$ and $\forall x,y\in X$. If $A(X)\bigcup B(X)\subseteq C(X)$ and $C(X)$ is a complete subspace of $X$. Then the maps $A,B\&C$ have a coincidence point in $X$. Moreover, if $(A,C)$ and $(B,C)$ are weakly compatible pairs, then $A,B\&C$ have a unique common fixed point in $X$.

**Proof:** Let $x_0$ be an arbitrary point in X and we define a sequence $\{y_k\}$ in X such that
$y_{2k} = Ax_{2k} = Cx_{2k+1}$ $\quad k = 0,1,2,3..........$
and $y_{2k+1} = Bx_{2k+1} = Cx_{2k+2}$ $\quad k = 0,1,2,3..........$
Consider that from (3.3.1) we have,

$$\int_0^{d(y_{2k},y_{2k+1})}\varphi(t)dt = \int_0^{d(Ax_{2k},Bx_{2k+1})}\varphi(t)dt$$

$$\le \alpha\int_0^{[d(Ax_{2k},Cx_{2k})+d(Bx_{2k+1},Cx_{2k+1})]}\varphi(t)dt + \beta\int_0^{[d(Cx_{2k},Cx_{2k+1})+d(Ax_{2k},Cx_{2k+1})+d(Bx_{2k+1},Cx_{2k})]}\varphi(t)dt$$

$$= \alpha\int_0^{[d(Ax_{2k},Bx_{2k-1})+d(Bx_{2k+1},Ax_{2k})]}\varphi(t)dt + \beta\int_0^{[d(Ax_{2k},Bx_{2k-1})+d(y_{2k},y_{2k+1})+d(y_{2k},y_{2k-1})]}\varphi(t)dt$$

$$\le \alpha\int_0^{[d(y_{2k},y_{2k-1})+d(y_{2k+1},y_{2k})]}\varphi(t)dt + \beta\int_0^{[d(y_{2k},y_{2k-1})+d(y_{2k},y_{2k+1})+d(y_{2k},y_{2k-1})]}\varphi(t)dt$$

$$\le \alpha\int_0^{[d(y_{2k},y_{2k-1})+d(y_{2k+1},y_{2k})]}\varphi(t)dt + \beta\int_0^{[2d(y_{2k},y_{2k-1})+d(y_{2k},y_{2k+1})]}\varphi(t)dt$$

$$\Rightarrow \int_0^{d(y_{2k},y_{2k+1})}\varphi(t)dt \le \frac{\alpha+2\beta}{1-\alpha-\beta}\int_0^{d(y_{2k},y_{2k-1})}\varphi(t)dt$$

$$\therefore \int_0^{d(y_{2k},y_{2k+1})}\varphi(t)dt \le h\int_0^{d(y_{2k},y_{2k-1})}\varphi(t)dt$$

where $h = \dfrac{\alpha+2\beta}{1-\alpha-\beta} < 1.$

Similarly we can show that

$$\int_0^{d(y_{2k+2},y_{2k+1})}\varphi(t)dt \le h'\int_0^{d(y_{2k+1},y_{2k})}\varphi(t)dt \text{ where } h' = \frac{\alpha+2\beta}{1-\alpha-\beta} < 1.$$

Therefore, for all $\forall k\in N$ we have,

$$\int_0^{d(y_{2k+2},y_{2k+1})}\varphi(t)dt \le h\int_0^{d(y_{2k},y_{2k+1})}\varphi(t)dt$$

$$\le h^2\int_0^{d(y_{2k},y_{2k+1})}\varphi(t)dt$$

$\le$…………………..
……………………
$\le h^{2k+1}d(y_0,y_1).$
Now for any m>k,

$$\int_0^{d(y_{2k},y_{2m})}\varphi(t)dt \le \int_0^{[d(y_{2k},y_{2k+1})+d(y_{2k+1},y_{2k+2})+............+d(y_{2m-1},y_{2m})]}\varphi(t)dt$$

$$\le h^{2k}\int_0^{d(y_0,y_1)}\varphi(t)dt + h^{2k+1}\int_0^{d(y_0,y_1)}\varphi(t)dt + ............ + h^{2m-1}\int_0^{d(y_0,y_1)}\varphi(t)dt$$

$$\le (h^{2k}+h^{2k+1}+h^{2k+2}.............)\int_0^{d(y_0,y_1)}\varphi(t)dt$$

$$= \frac{h^{2k}}{1-h}\int_0^{d(y_0,y_1)}\varphi(t)dt.$$

Since P is normal with normal constant k.

Therefore, $\left\|\int_0^{d(y_{2k},y_{2m})}\varphi(t)dt\right\| \le \frac{h^{2k}}{1-h}k\left\|\int_0^{d(y_0,y_1)}\varphi(t)dt\right\|.$

Then, $\int_0^{d(y_{2k},y_{2m})}\varphi(t)dt \to 0$ as $k,m\to\infty$ with $0<\alpha+2\beta<1$ so $\{y_{2k}\}=\{Cx_{2k}\}$ is a Cauchy sequence.

Now, since C(X) is a complete subspace of X, so $\exists v\in C(X)$ such that

$\lim_{n\to\infty} y_{2k} = v$ i.e. $Cx_{2k}\to v$ as $k\to\infty.$

Consequently, we can find u in X such that $C(u)=v.$

We shall show that $Cu = Au = Bu.$

For this we consider,

$$d(Au,y_{2k+1}) = d(Au,Bx_{2k+1})$$

$$\le \alpha\int_0^{[d(Au,Cu)+d(Bx_{2k+1},Cx_{2k+1})]}\varphi(t)dt + \beta\int_0^{[d(Cu,Cx_{2k+1})+d(Au,Cx_{2k+1})+d(Bx_{2k+1},Cu)]}\varphi(t)dt$$

$$\le \alpha\int_0^{[d(Cx_{2k+1},Cu)+d(Au,Cx_{2k+1})]}\varphi(t)dt + \beta\int_0^{[d(Cu,Cx_{2k+1})+d(Au,y_{2k})+d(y_{2k+1},Cu)]}\varphi(t)dt.$$

$$\le \alpha\int_0^{[d(Cx_{2k+1},Cu)+d(Bx_{k+1},Au)]}\varphi(t)dt + \beta\int_0^{[d(Cu,Cx_{2k+1})+d(Au,y_{2k})+d(Cx_{2k+1},Cu)]}\varphi(t)dt$$

$$\Rightarrow \int_0^{d(Au,y_{2k+1})}\varphi(t)dt \le \frac{\alpha+2\beta}{1-\alpha-\beta}\int_0^{d(Cx_{2k+1},Cu)}\varphi(t)dt$$

Since $P$ is a normal cone with normal constant $k$, so

$$\left\|\int_0^{d(Au,y_{2k+1})}\varphi(t)dt\right\| = \left\|\int_0^{d(Au,Bx_{2k+1})}\varphi(t)dt\right\| \le k.\frac{\alpha+2\beta}{1-\alpha-\beta}\left\|\int_0^{d(Cx_{2k+1},Cu)}\varphi(t)dt\right\| \to 0 \text{ as } k\to\infty$$

Therefore, for larger $n$ we get

$$\int_0^{d(Cu,Au)}\varphi(t)dt \le \int_0^{d(v,Cx_{2k+2})}\varphi(t)dt \le \int_0^{d(v,u)}\varphi(t)dt = 0$$

Implies that $\int_0^{d(Cu,Au)}\varphi(t)dt = 0$ and hence $Cu = v = Au.$

Similarly we can show that

$Cu = v = Bu$ therefore we have,

$v = Cu = Au = Bu$.

i.e. $u$ is a coincidence point of mappings $A,B,C$.

Now we will show that $A,B,C$ have a unique coincidence point.

For this we assume that there exists another point $x^*\in X$ such that,

$Ax^* = Bx^* = Cx^* = y^*$ for some $x^*\in X$.

Then we have,

$$\int_0^{d(y^*,v)}\varphi(t)dt = \int_0^{d(Ax^*,Bu)}\varphi(t)dt$$

$$\le \alpha\int_0^{[d(Au,Cu)+d(Bx_{2k+1},Cx_{2k+1})]}\varphi(t)dt + \beta\int_0^{[d(Cu,Cx_{2k+1})+d(Au,Cx_{2k+1})+d(Bx_{2k+1},Cu)]}\varphi(t)dt$$

$$\le \alpha\int_0^{[d(y^*,y^*)+d(v,v)]}\varphi(t)dt + \beta\int_0^{[d(y^*,Cu)+d(y^*,Cu)+d(Bu,y^*)]}\varphi(t)dt$$

$$= 3\beta\int_0^{[d(y^*,v)+d(y^*,v)+d(v,y^*)]}\varphi(t)dt$$

$\Rightarrow \int_0^{d(y^*,v)}\varphi(t)dt \le 3\beta\int_0^{d(y^*,v)}\varphi(t)dt$ which gives a contraction.

Hence $\left\|\int_0^{d(y^*,v)}\varphi(t)dt\right\| = 0.$ so we have $y^* = v.$

Thus the point of coincidence is unique.

If pairs (A, C) and (B, C) are weakly compatible then

$Av = ACu = CAu = Cv$

And $Bv = BCu = CBu = Cv.$

Therefore, $Av = Bv = Cv = w$ (say). This shows that w is another point of coincidence. Therefore by uniqueness, we must have

w = v i.e. $Av = Bv = Cv = v.$

Thus v is a unique common fixed point of self maps A, B and C.

This completes the proof of the theorem.

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