The planarity of bipartite graphs in R

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Abstract

Let R be a commutative ring and let Z(R) be its set of zero-divisors. We associate a graph Γ(R) to R with vertices Z(R)∗ = Z(R) - {0}, the set of non-zero zero divisors of R and for distinct u, v ∈ Z(R)∗, the vertices u and v are adjacent if and only if uv = 0. In this paper, we evaluate the consistency of rectilinear crossing number of complete bipartite zero divisor graphs, in which the transformation of a non-planar graph into a planar graph is obtained by framing formula using removal of edges and removal of crossings.

Keywords: Rectilinear crossing number, planar graph, zero divisor graph

Introduction

A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a Planar graph, and such a drawing is called a Planar embedding of the graph. Let G be a graph drawn in the plane with the requirement that the edges are line segments, no three vertices are collinear, and no three edges may intersect in a point, unless the point is a vertex. Such a drawing is said to be a Rectilinear drawing of G. The rectilinear crossing number of G, denoted cr(G), is the fewest number of edge crossings attainable over all rectilinear drawings of G [3]. Any such a drawing is called optimal. The idea of a zero divisor graph of a commutative ring was introduced by I. Beck in [1]. The zero divisor graph is very useful to find the algebraic structures and properties of rings. We mainly focus on D. F. Anderson and P. S. Livingston’s zero divisor graphs [2].

Basic Definitions

Definition - 1
If a and b are two non-zero elements of a ring Z_p such that a.b = 0, then ‘a’ and ‘b’ are the Zero divisors of commutative ring Z_p.

Definition - 2
If a graph G = (V, E) is a maximum planar subgraph of a graph G = (V, E) such that there is no planar subgraph G′ = (V, E′) of G with |E′| > |E|, then G′ is called a maximum planar subgraph of G.

The Planarity of Complete Bipartite graphs

Theorem - 1
If p and q are distinct prime numbers with q > p, then,
\[ \text{cr} \left( \Gamma(Z_{pq}) \right) = (p - 1)(p - 3)(q - 1)(q - 3)/16 \, [10]. \]

Theorem - 2
If p and q are distinct prime numbers with q > p, then, the consistency of Rectilinear crossing of \( \text{cr} \left( \Gamma(Z_{pq}) \right) \),

(i) When removing the edges, the edge planarity is,
\[ P \left[ E \left( \Gamma(Z_{pq}) \right) \right] = pq - 3(p + q) + 9 \]

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(ii) When removing the edges involved in crossings then, 

\[
P \left[ c \overline{r} \left( I' \left( Z_{pq} \right) \right) \right] = n \left[ E \left[ c \overline{r} \left( I' \left( Z_{pq} \right) \right) \right] \right] - n \left[ Pair \left( c \overline{r} \left( I' \left( Z_{pq} \right) \right) \right) \right]
\]

**Proof:**

The vertex set of \( I' \left( Z_{pq} \right) \) is \( V \left( I' \left( Z_{pq} \right) \right) = \{ p, 2p, \ldots, p(q-1), q, 2q, \ldots, (p-1)q \} \). Then \( |V \left( I' \left( Z_{pq} \right) \right)| = p + q - 2 \).

By the above theorem \( I' \left( Z_{pq} \right) \) is a bipartite graph \( K_{p-1,q-1} \). The edge set \( E \left( I' \left( Z_{pq} \right) \right) \) is,

\[
\begin{align*}
(p, q), & \ldots, (p, (p-1)q) \\
(2p, q), & \ldots, (2p, (p-1)q) \\
\vdots & \vdots \\
(p(q-1), q), & \ldots, (p(q-1), q(p-1))
\end{align*}
\]

Then \( |E \left( I' \left( Z_{pq} \right) \right)| = (p-1)(q-1) \)

To find the complete planarity of \( I' \left( Z_{pq} \right) \) and thereby finding the consistency of the graph:

i. By removal of edges

ii. By removal of crossings

**The complete planarity of \( I' \left( Z_{pq} \right) \) by removal of edges**

Let the complete planar graph, after the removal of edges is denoted by, \( P \left[ E \left( I' \left( Z_{pq} \right) \right) \right] \). The total number of edges is denoted by

\[
n \left[ E \left( I' \left( Z_{pq} \right) \right) \right] = pq - (p + q) + 1.
\]

The number of edges that are involved in crossings is denoted by \( n \left[ E_1 \left( I' \left( Z_{pq} \right) \right) \right] = 2[(p - 1) = (q - 1)] \).

The number of edges that are not involved in crossings is denoted by \( n \left[ E_2 \left( I' \left( Z_{pq} \right) \right) \right] = 4 \) for all bipartite graphs.

Therefore we have, \( P \left[ E \left( I' \left( Z_{pq} \right) \right) \right] \)

\[
= n \left[ E \left( I' \left( Z_{pq} \right) \right) \right] - n \left[ E_1 \left( I' \left( Z_{pq} \right) \right) \right] - n \left[ E_2 \left( I' \left( Z_{pq} \right) \right) \right]
\]

\[
= pq - (p + q) + 1 - 2(p - 1 + q - 1) + 4
\]

\[
= pq - 3(p + q) + 9
\]

The complete planarity of \( I' \left( Z_{pq} \right) \) by removal of crossings:

Since the crossings of the edges involve pair of crossing, we proceed by defining \( Pair \left[ c \overline{r} \left( V_1, V_2 \right) \right] \) and for the vertex set, \( V_1 = \{ p, 2p, \ldots, p(q-1) \} \) and \( V_2 = \{ q, 2q, \ldots, q(p-1) \} \).

we find the crossings between \( V_1 \) and \( V_2 \) denoted by \( Pair \left[ c \overline{r} \left( V_1, V_2 \right) \right] \)

\[
\begin{align*}
\overline{r}(p, q), & \ldots, \overline{r}(p, (p-1)q) \\
\overline{r}(2p, q), & \ldots, \overline{r}(2p, (p-1)q) \\
\vdots & \vdots \\
\overline{r}(p(q-1), q), & \ldots, \overline{r}(p(q-1), q(p-1))
\end{align*}
\]

Summing up the crossings, \( Pair \left[ c \overline{r} \left( V_1, V_2 \right) \right] = \sum_{m=1}^{(q-1)} \sum_{n=1}^{(p-1)} (mp, nq) \)

We observe that there are \( \frac{(p-1)(p-3)}{2} \sum_{n=1}^{q-3} n \) crossings. \( n \left[ Pair \left[ c \overline{r} \left( V_1, V_2 \right) \right] \right] \)

\[
= n \left[ Pair \left( E \left( c \overline{r} \left( I' \left( Z_{pq} \right) \right) \right) \right) \right]
\]

From theorem 1, total edge crossings is,

\[
n \left[ E \left( c \overline{r} \left( I' \left( Z_{pq} \right) \right) \right) \right] = (p - 1)(p - 3)(q - 1)(q - 3)/16
\]

Removing \( n \left[ Pair \left( E \left( c \overline{r} \left( I' \left( Z_{pq} \right) \right) \right) \right) \right] \), we get the complete planarity as,

\[
P \left[ c \overline{r} \left( I' \left( Z_{pq} \right) \right) \right] = n \left[ E \left( c \overline{r} \left( I' \left( Z_{pq} \right) \right) \right) \right] - n \left[ Pair \left( E \left( c \overline{r} \left( I' \left( Z_{pq} \right) \right) \right) \right) \right]
\]
\[
= \left( \frac{p-1}{2} \right) \left( \frac{p-3}{2} \right) \left( \frac{q-1}{2} \right) \left( \frac{q-3}{2} \right) - \frac{(p-1)(p-3)}{2} \sum_{n=1}^{q-3} n
\]

So, let us prove this by induction in \(p\) and \(q\), \(p < q\). When \(p = 3, q = 5\) the graph \(K_{2,q}\) is a planar graph which is trivial. So we proceed for the following cases

Case (i): Let \(p = 5\)

Subcase (i): Let \(q = 7\)

The vertex set of \(\Gamma(g_{pq})\) is \(V(\Gamma(Z_{35})) = \{5, 10, \ldots, 5(q-1), q, 2q, 3q, 4q\}\). Let \(V_1 = \{5, 10, \ldots, 5(q-1)\}\) and \(V_2 = \{q, 2q, 3q, 4q\}\). The edge set of \(\Gamma(Z_{35})\) is \(E(\Gamma(Z_{35})) = \{(5, q) \ldots (5, 4q), (10, q) \ldots (10, 4q)\}\)

The complete planarity of \(\Gamma(Z_{35})\) by removal of edges:

\[
E(\Gamma(Z_{35})) = 4(6) - 2(46) + 4 = (p - 1)(q - 1) - 2(p - 1 + q - 1) + 4 = pq - 3(p + q) + 9
\]

Therefore by removing 8 edges the graph \(\Gamma(Z_{35})\) becomes planar.

The complete planarity of \(\Gamma(Z_{35})\) by removal of crossings:

\[
\text{Pair} \left[ \tilde{\Gamma}(\Gamma(Z_{35})) \right] = \begin{bmatrix} 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}
\]

Summing up the crossings,

\[
\text{Pair} \; \tilde{c}\Gamma(V_1, V_2) = \sum_{m=1}^{6} \sum_{n=1}^{4} (5m, 7n)
\]

\[
= \text{Pair} \; \tilde{c}\Gamma \left( \frac{(5, 7) + \cdots + (5, 28) + (10, 7) + \cdots + (10, 28)}{+15(7) + \cdots + (15, 28) + (20, 7) + \cdots + (20, 28)} + (25, 7) + \cdots + (25, 28) + (30, 7) + \cdots + (30, 28) \right)
\]

\[
= 2[(2 + 0 + 0 + 2) + (1 + 0 + 0 + 1) + (0 + 0 + 0 + 0)] = 4(2 + 1)
\]

\[
= \frac{(p - 1)(p - 3)}{2} \sum_{n=1}^{q-3} n
\]

From theorem 1, \(n \left[ E \left( \tilde{\Gamma}(\Gamma(Z_{35})) \right) \right] = 12\)

Therefore \(P[\tilde{c}\Gamma(\Gamma(Z_{35}))]\)

\[
= n \left[ E \left( \tilde{\Gamma}(\Gamma(Z_{35})) \right) \right] - n \left[ \text{Pair} \left( E \left( \tilde{\Gamma}(\Gamma(Z_{35})) \right) \right) \right]
\]

\[
= 0 = 12 - 12 = (2)(1)(3)(2) - 4(2 + 1)
\]

\[
= \left( \frac{5 - 1}{2} \right) \left( \frac{5 - 3}{2} \right) \left( \frac{7 - 1}{2} \right) \left( \frac{7 - 3}{2} \right) - 4(2 + 1)
\]
\[ \left( \frac{p-1}{2} \right) \left( \frac{p-3}{2} \right) \left( \frac{q-1}{2} \right) \left( \frac{q-3}{2} \right) - \frac{(p-1)(p-3)}{2} \sum_{n=1}^{q-3} n \]

Case (ii): Let \( p = 7 \)

Subcase (i): Let \( q = 11 \)

The vertex set of \( \Gamma(Z_{pq}) \) is \( V(\Gamma(Z_{77})) = \{7, 14, \ldots, 7(q - 1), q, 2q, \ldots, 6q\} \). Let \( V_1 = \{7, 14, \ldots, 7(q - 1)\} \) and \( V_2 = \{q, 2q, \ldots, 6q\} \). The edge set of \( \Gamma(Z_{77}) \) is

\[ E(\Gamma(Z_{77})) = \{(7, q) \ldots (7, 6q), (14, q) \ldots (14, 6q) \} \]

The complete planarity of \( \Gamma(Z_{77}) \) by removal of edges:

\[ n[E(\Gamma(Z_{77}))] = 60, n[E_1(\Gamma(Z_{77}))] = 32 \]

and \( n[E_2(\Gamma(Z_{77}))] = 4 \). Therefore, \( P[E(\Gamma(Z_{77}))] = n[E(\Gamma(Z_{77}))] - n[E_1(\Gamma(Z_{77}))] - n[E_2(\Gamma(Z_{77}))] \)

\[ = 60 - 32 - 4 = 60(10) - 2(6 + 10) + 4 \]

\[ = (p - 1)(q - 1) - 2(p - 1 + q - 1) + 4 \]

\[ = pq - 3(p + q) + 9 \]

Therefore by removing 32 edges the graph \( \Gamma(Z_{77}) \) becomes planar.

The complete planarity of \( \Gamma(Z_{77}) \) by removal of crossings

\[ \text{Pair}[\overline{\text{c}\,\text{r}}(\Gamma(Z_{77}))] = \begin{bmatrix} 0 & 1 & 2 & 4 & 6 & 8 & 8 & 6 & 4 & 2 & 0 \\ 0 & 1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 4 & 3 & 2 & 1 & 0 \\ 0 & 2 & 4 & 6 & 8 & 8 & 6 & 4 & 2 & 0 \end{bmatrix} \]

Summing up the crossings,

\[ \text{Pair} \, \overline{\text{c}\,\text{r}}(V_1, V_2) = \sum_{m=1}^{10} \sum_{n=1}^{6} (7m, 11n) \]

\[ \text{Pair} \, \overline{\text{c}\,\text{r}} \left\{ \begin{array}{l} (7, 11) + \cdots + (7, 66) + \\ (14, 11) + \cdots + (14, 66) + \\ \cdots + \\ (77, 11) + \cdots + (77, 66) \end{array} \right\} \]

\[ = 2[(2 + 1 + 0 + 0 + 1 + 2) + (4 + 1 + 2 + 0 + 0 + 2 + 4)] + 2[(6 + 3 + 0 + 0 + 3 + 6) + (8 + 4 + 0 + 0 + 4 + 8)] \]

\[ = 6(2)(1 + 2 + 3 + 4) \]

\[ = \frac{(p - 1)(p - 3)}{2} \sum_{n=1}^{q-3} n \]

From theorem 1, \( n[E(\overline{\text{c}\,\text{r}}(\Gamma(Z_{77})))] = 120 \)

Therefore \( P[\overline{\text{c}\,\text{r}}(\Gamma(Z_{77}))] \)

\[ = n[E(\overline{\text{c}\,\text{r}}(\Gamma(Z_{77})))] - n[\text{pair} \left( E(\overline{\text{c}\,\text{r}}(\Gamma(Z_{77}))) \right)] \]

\[ = 0 = 120 - 120 = (3)(2)(5)(4) - 6(2)(1 + 2 + 3 + 4) \]

\[ = \left( \frac{7 - 1}{2} \right) \left( \frac{7 - 3}{2} \right) \left( \frac{11 - 1}{2} \right) \left( \frac{11 - 3}{2} \right) - 12(1 + 2 + 3 + 4) \]
\[
\sum_{n=1}^{q-3} \left( \frac{p-1}{2} \right) \left( \frac{p-3}{2} \right) \left( \frac{q-1}{2} \right) \left( \frac{q-3}{2} \right) - \frac{(p-1)(p-3)}{2} \]

The Planarity of Bipartite graphs

**Theorem-3:**
For any graph, \( \Gamma(Z_{pqr}) \) where \( p = 2; q = 3 \) and \( r > 3 \) then \( cr(\Gamma(Z_{pqr})) = cr(\Gamma(Z_{pqr})) + (r - 1)/2 \). [9, 10]

**Theorem-4:**
For any graph \( \Gamma(Z_{pqr}) \), \( p=2,q=3 \) and \( r > 3 \), the consistency of Rectilinear crossing of \( \overline{cr}(\Gamma(Z_{pqr})) \).

(i) When removing the edges, the edge planarity is,
\[
P \left[ E \left( \Gamma(Z_{pqr}) \right) \right] = \frac{pq}{2} (r - 3) + (p - 1)
\]

(ii) When removing the edges involved in crossings then,
\[
P \left[ cr(\Gamma(Z_{pqr})) \right] = n \left[ E \left[ \overline{cr}(\Gamma(Z_{pqr})) \right] \right] - n \left[ Pair \left[ \overline{cr}(\Gamma(Z_{pqr})) \right] \right]
= (r - 1)(r - 2) - [(r - 1)(r - 4) + (r - 1) + (r - 1)]
\]

**Proof**
The vertex set of \( \Gamma(Z_{pqr}) \) is \( V(\Gamma(Z_{pqr})) = \{ p, 2p, \ldots q(pr - 1), r, 2r, \ldots, 5r \} \). Then \( V(\Gamma(Z_{pqr})) = 2p(r - 1) + (pq - 1) \) As \( \Gamma(Z_{pqr}) \) is a bipartite graph [6] and can be decomposed as \( K_{p-1,q-1} + K_{p-1,r-1} + K_{p-1,2(r-1)} + K_{q-1,2(r-1)} + K_{q-1,r-1} \). The Rectilinear drawing of \( \Gamma(Z_{pqr}) \) follows from theorem 4.

The edge set \( E(\Gamma(Z_{pqr})) \) can be obtained from the following split vertex sets. Let

\[ V_1 = \{ r, 2r, \ldots (pq - 1)r \}, \quad V_2 = \{ q, 3q, \ldots q(pr - 1) \}, \quad V_3 = \{ p, 2p, 4p, 7p, \ldots q(pr - 1) \}, \quad V_4 = \{ pq, 2pq, \ldots (r - 1)pq \} \]

Then the edge set of \( E(\Gamma(Z_{pqr})) \) can be split for convenience as follows.

\[ E_1(\Gamma(Z_{pqr})) = \{(3r, 4r)\} \]
\[ E_2(\Gamma(Z_{pqr})) = \{(3r, p) \ldots (3r, p(qr - 1))\} \]
\[ E_3(\Gamma(Z_{pqr})) = \{(3r, 2r)\} \]
\[ E_4(\Gamma(Z_{pqr})) = \{(3r, pq) \ldots (3r, (r - 1)pq)\} \]
\[ E_5(\Gamma(Z_{pqr})) = \{E(\Gamma(Z_{pqr})) - E_1 - E_2 - E_3 - E_4\} \]

To find the complete planarity of \( \Gamma(Z_{pqr}) \) and thereby finding the consistency of the graph:
(i) By removal of edges
(ii) By removal of crossings

The complete planarity of \( \Gamma'(Z_{pqr}) \) by removal of edges:

Let the complete planar graph, after the removal of edges is denoted by \( P \left( E(\Gamma'(Z_{pqr})) \right) \). The total number of edges in \( \Gamma'(Z_{pqr}) \) is denoted by \( n \left( E(\Gamma'(Z_{pqr})) \right) = q^2(r - 1) + p \). The number of edges that are involved in Rectilinear crossings in \( \Gamma'(Z_{pqr}) \) are denoted by \( n \left( E_3(\Gamma'(Z_{pqr})) \right) \) and \( n \left( E_4(\Gamma'(Z_{pqr})) \right) \) and are equal to \( (p - 1) \) and \( \frac{pq}{2} (r - 3) \) respectively. The number of edges that are not involved in Rectilinear crossings are denoted and are represented by \( n \left( E_1(\Gamma'(Z_{pqr})) \right) = 1 = (p - 1) \), \( n \left( E_2(\Gamma'(Z_{pqr})) \right) = 2(r - 1) \).

\[
\begin{align*}
= n \left[ E_1(\Gamma'(Z_{pqr})) \right] - n \left[ E_2(\Gamma'(Z_{pqr})) \right] - n \left[ E_3(\Gamma(Z_{pqr})) \right] \\
= q^2(r - 1) + p(p - 1) - 2(r - 1)q^2(r - 1) + 2(r - 1) + \frac{pq}{2} (r - 3) = \frac{pq}{2} (r - 3) + 1
\end{align*}
\]

Therefore removing \( \frac{pq}{2} (r - 3) + 1 \) edges from \( E(\Gamma'(Z_{pqr})) \), a complete planar graph is obtained.

The complete planarity of \( \Gamma'(Z_{pqr}) \) by removal of crossings

The crossings between every two edges involve pair of crossing. So we make the calculation simple by defining Pair-crossing matrix denoted by \( Pair \left( \bar{c}r(\Gamma'(Z_{pqr})) \right) \). Now we find the rectilinear crossing between the vertices in the vertex sets \( V_1 \) and \( V_2 \cup V_4 = V_1' \) (say). Therefore the vertices in the vertex set, will be as follows.

\[
V_1' = \{p, 2p, ... p(qr - 1), q, 2q, ... q(pr - 1)\}
\]

So we proceed by finding the crossings between \( V_1 \) and \( V_1' \) denoted by \( Pair \left( \bar{c}r(V_1, V_1') \right) \) and defined as,

\[
\begin{array}{cccc}
\bar{c}r(3r, 2r) & (q, r) & (q, r) & (q, r) \\
\bar{c}r(3r, p) & (q, r) & (q, r) & (q, r) \\
\bar{c}r(3r, 4r) & (q, r) & (q, r) & (q, r) \\
\bar{c}r(3r, 5r) & (q, r) & (q, r) & (q, r) \\
\bar{c}r(5r, 2q) & (q, r) & (q, r) & (q, r) \\
\bar{c}r(5r, 3q) & (q, r) & (q, r) & (q, r) \\
\bar{c}r(5r, 4q) & (q, r) & (q, r) & (q, r) \\
\bar{c}r(5r, 5q) & (q, r) & (q, r) & (q, r) \\
\end{array}
\]

where the edges with bar represents that the vertices are non-adjacent. So we find the rectilinear crossings of the remaining edges. Now splitting the edge set \( E_4(\Gamma'(Z_{pqr})) \) for convenience as follows.

\[
E_4' \left( \Gamma(Z_{pqr}) \right) = \{(3r, pq), ... (3r, (r - 1)pq)\}
\]

\[
E_4'' \left( \Gamma(Z_{pqr}) \right) = \{(5r, pq), ... (5r, (r - 1)pq)\}
\]

\[
Pair \bar{c}r \left( E_4' \left( \Gamma(Z_{pqr}) \right) \right) = (r - 1)(r - 4)
\]

\[
Pair \bar{c}r \left( E_4'' \left( \Gamma(Z_{pqr}) \right) \right) = (r - 1)
\]

Combining we get,

\[
n[Pair \bar{c}r(V_1, V_1')] = (r - 1)(r - 4) + (r - 1) + (r - 1)
\]

\[
= n \left[ Pair \bar{c}r \left( E(\Gamma(Z_{pqr})) \right) \right]
\]

Since from theorem 3, total edge crossings is,

\[
n \left[ \bar{c}r \left( E(\Gamma(Z_{pqr})) \right) \right] = (r - 1)(r - 2)
\]

Removing \( n \left[ Pair \bar{c}r \left( E(\Gamma(Z_{pqr})) \right) \right] \) we get the complete planarity as \( P \left[ \bar{c}r \left( \Gamma(Z_{pqr}) \right) \right] \)
From theorem 3 the rectilinear crossing number of 
$\overline{\text{Pair}}(\Gamma'(Z_{pq}))$ is 

$= n \left[ E \left[ \overline{\text{cr}} \left( \Gamma'(Z_{pq}) \right) \right] \right] - n \left[ \text{Pair} \left[ \overline{\text{cr}} \left( \Gamma'(Z_{pq}) \right) \right] \right]$ 

So let us prove this by induction on $p$, $q$ and $r$, where $p < q < r$. Now consider for the case $p = 2, q = 3$ and $r > q$

Case (i): Let $p = 2, q = 3$

Subcase (i): Let $r = 5$

The vertex set of $\Gamma'(Z_{30})$ is, 
$V(\Gamma'(Z_{30})) = \{2,4,28,3,9,27,5,10,25\}$. Let $V_1 = \{5,10,15,20,25\}, V_2 = \{3,9,21,27\}, V_3 = \{2,4,8,28\}$

and $V_4 = \{6,12,18,24\}$. The edge set of $\Gamma'(Z_{30})$ is

$E(\Gamma'(Z_{30})) = \begin{cases} 
(5,6) \ldots (5,24), (10,3) \ldots (10,27), (10,6) \ldots (10,24) \\
(15,2) \ldots (15,28), (15,6) \ldots (15,24), (15,10), (15,20) \\
(20,3) \ldots (20,27), (20,6) \ldots (20,24), (25,6) \ldots (25,24) 
\end{cases}$

The complete planarity of $\Gamma'(Z_{30})$ by removal of edges:

$n [E(\Gamma'(Z_{30}))] = 38 = q^2(r - 1),$

$n [E_1(\Gamma'(Z_{30}))] = \{(15,20)\} = 1 = (p - 1) $

$n [E_2(\Gamma'(Z_{30}))] = \{(15,2), \ldots (15,28)\} = 8 = 2(r - 1) $

$n [E_3(\Gamma'(Z_{30}))] = \{(15,10)\} = 1 = (p - 1) $

$n [E_4(\Gamma'(Z_{30}))] = \{(5,6), (5,24), (15,6), (15,24), (25,6), (25,24)\}$

$= 6 = \frac{pq}{3} (r - 3) $

$n [E_5(\Gamma'(Z_{30}))] = n[E(\Gamma'(Z_{30}))] - n[E_1(\Gamma'(Z_{30}))] - n[E_2(\Gamma'(Z_{30}))] - n[E_3(\Gamma'(Z_{30}))] - n[E_4(\Gamma'(Z_{30}))]$ 

$= 22 = q^2(r - 1) - 2(r - 1) - \frac{pq}{2} (r - 3)$

where $n[E_3(\Gamma'(Z_{30}))]$ and $n[E_4(\Gamma'(Z_{30}))]$ represents number of edges that are involved in crossings and the remaining represents the number of edges that are not involved in crossings. Therefore $P[E(\Gamma'(Z_{30}))]$

$= n[E(\Gamma'(Z_{30}))] - n[E_1(\Gamma'(Z_{30}))] - n[E_2(\Gamma'(Z_{30}))] - n[E_3(\Gamma'(Z_{30}))] = 38 - (22 + 8 + 1) = 7 = \frac{pq}{2} (r - 3)$

Therefore by removing 7 edges the graph $\Gamma'(Z_{30})$ becomes planar.

The complete planarity of $\Gamma'(Z_{30})$ by removal of crossings

Pair cross matrix of $\overline{\text{cr}}(\Gamma'(Z_{30}))$ is obtained from the rectilinear crossing between the vertex sets $V_1$ and $V_2 \cup V_4 = V_1'$ (say).

$\text{Pair}(\overline{\text{cr}}(V_1, V_1')) = \begin{bmatrix} 
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix} + \overline{\text{cr}}(3r, 2r)$

Now splitting the edge set $E_4'(\Gamma'(Z_{30}))$ for convenience as follows. $E_4'(\Gamma'(Z_{30})) = \{(15,6), (15,24), (5,6), (5,24)\}$

$E_4''(\Gamma'(Z_{30})) = \{(25,6), (25,24)\}$

$\text{Pair} \overline{\text{cr}}[E_4'(\Gamma'(Z_{30}))] = (4)(1) = (5 - 1)(5 - 4) = (r - 1)(r - 4)$

$\text{Pair} \overline{\text{cr}}[E_4''(\Gamma'(Z_{30}))] = 4 = (5 - 1) = (r - 1)$

$\text{Pair} \overline{\text{cr}}[e_3(\Gamma'(Z_{30}))] = \overline{\text{cr}}(15,10) = 4 = (r - 1)$

Combining we get, $n[\text{Pair} \overline{\text{cr}}(V_1, V_1')] = 12 = 4 + 4 + 4$

$= (r - 1)(r - 4) + q(r - 1) + (r - 1) = n[\text{Pair} \overline{\text{cr}}(E(\Gamma'(Z_{30})))]

$From theorem 3 the rectilinear crossing number of $\Gamma'(Z_{30})$ is

$\overline{\text{cr}}(\Gamma'(Z_{30})) = 12 = (r - 1)(r - 2)$

Removing $n[\text{Pair} \overline{\text{cr}}(E(\Gamma'(Z_{30})))], we get the complete planarity. That is, $P[\overline{\text{cr}}(\Gamma'(Z_{30}))]\n
= n[E(\overline{\text{cr}}(\Gamma'(Z_{30})))] - n[\text{Pair} \overline{\text{cr}}(\Gamma'(Z_{30})))]

= 0 = 12 - 12 = 4(3) - (4 + 4 + 4)$

$= (r - 1)(r - 2) - [(r - 1)(r - 4) + (r - 1) + (r - 1)]$
Subcase (ii): Let \( r = 7 \)
The vertex set of \( \Gamma(Z_{42}) \) is,

\[
V(\Gamma(Z_{42})) = \{2, 4, 30, 36, 9, 39, 7, 14, 35\}. \text{Let } V_1 = \{7, 14, \ldots, 35\}, V_2 = \{3, 9, 15, 27, 33, 39\}, V_3 = \{2, 4, 8, \ldots, 40\} \text{ and } V_4 = \{6, 12, \ldots, 36\}. \text{ The edge set of } \Gamma(Z_{42}) \text{ is}
\]

\[
E(\Gamma(Z_{42})) = \left\{\begin{array}{c}
(7, 6), (7, 36), (14, 3), (14, 39), (14, 6), (14, 36) \\
(21, 6), (21, 24), (21, 14), (21, 28) \\
(28, 3), (28, 39), (28, 6), (28, 36), (35, 6), (35, 36)
\end{array}\right\}
\]

The complete planarity of \( \Gamma(Z_{30}) \) by removal of edges

\[
n\left|E(\Gamma(Z_{42}))\right| = 56 = q^2(r - 1)
\]

\[
n\left|E_3(\Gamma(Z_{42}))\right| = (21, 28) = 1 = (p - 1)
\]

\[
n\left|E_4(\Gamma(Z_{42}))\right| = (21, 14) = 1 = (p - 1)
\]

\[
n\left|E_4(\Gamma(Z_{42}))\right| = ((21, 6), (21, 42), (7, 6), (7, 42), (35, 6), (35, 42))
\]

\[
= 30 = q^2(r - 1) - 2(r - 1) - \frac{pq}{2}(r - 3)
\]

where \( n[E_3(\Gamma(Z_{42}))] \) and \( n[E_4(\Gamma(Z_{42}))] \) represents number of edges that are involved in crossings and the remaining represents the number of edges that are not involved in crossings. Therefore \( P[E(\Gamma(Z_{42}))] \)

\[
= n\left|E(\Gamma(Z_{42}))\right| - n\left|E_3(\Gamma(Z_{42}))\right| - n\left|E_4(\Gamma(Z_{30}))\right| - n\left|E_4(\Gamma(Z_{42}))\right| = 30(30 + 12 + 1) = 13 = \frac{pq}{2}(r - 3)
\]

Therefore by removing 13 edges the graph \( \Gamma(Z_{42}) \) becomes planar.

The complete planarity of \( \Gamma(Z_{42}) \) by removal of crossings:

Pair cross matrix of \( \tilde{\Gamma}(\Gamma(Z_{42})) \) is obtained from the rectilinear crossing between the vertex sets \( V_1 \) and \( V_2 \cup V_4 = V_1' \) (say).

\[
Pair[\tilde{\Gamma}(V_1, V_1')] = \begin{bmatrix}
0 & 0 & 3 & 2 & 0 & 0 & 2 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 0
\end{bmatrix} + \tilde{\Gamma}(3r, 2r)
\]

Now splitting the edge set \( E_4'(\Gamma(Z_{42})) \) for convenience as follows. \( E_4'(\Gamma(Z_{42})) = ((21, 6), (21, 42), (7, 6), (7, 42)) \)

\[
E_4''(\Gamma(Z_{42})) = ((35, 6), (35, 42))
\]

\[
Pair[\tilde{\Gamma}E_4'(\Gamma(Z_{42}))] = 18 = 6(3) = (7 - 1)(7 - 4) = (r - 1)(r - 4)
\]

\[
Pair[\tilde{\Gamma}E_4''(\Gamma(Z_{42}))] = 6 = (7 - 1) = (r - 1)
\]

\[
Pair[\tilde{\Gamma}E_4(\Gamma(Z_{42}))] = \tilde{\Gamma}(21, 14) = 6 = (r - 1)
\]

Combining we get, \( n[\text{Pair}[\tilde{\Gamma}(V_1, V_1')] \] = 30 = 18 + 6 + 6

\[
= (r - 1)(r - 4) + (r - 1) + (r - 1) = n[\text{Pair}[\tilde{\Gamma}(E(\Gamma(Z_{42})))]]
\]

From theorem 3 the rectilinear crossing number of \( \Gamma(Z_{42}) \) is

\[
\tilde{\Gamma}(\Gamma(Z_{42})) = 30 = (r - 1)(r - 2)
\]

Removing \( n[\text{Pair}[\tilde{\Gamma}(E(\Gamma(Z_{42})))]] \), we get the complete planarity. That is, \( P[\tilde{\Gamma}(\Gamma(Z_{42}))] \)

\[
= n\left[\tilde{\Gamma}(E(\Gamma(Z_{42})))\right] - n[\text{Pair}[\tilde{\Gamma}(\Gamma(Z_{42}))]]
\]

\[
= 0 = 30 - 30 = 4(3) - (4 + 4 + 4)
\]

\[
= (r - 1)(r - 2) - [(r - 1)(r - 4) + (r - 1) + (r - 1)]
\]

Conclusion

In this paper we find a maximum planar subgraph from a complete bipartite graphs, especially for zero divisor graphs in any rectilinear drawing of G. Note that for any zero divisor graph \( \Gamma(Z_{pq}) \), we get a complete bipartite graph \( K_{p-1,q-1} \) where \( p \) and \( q \) are primes. Therefore a complete bipartite graph is formed with even number of vertices and we suggest any bipartite or complete bipartite graph can be made into a maximum planar graph with odd number of vertices for future work. We infer from the above formulae that the removal of edges involved in crossings leading to a planar graph can be applied in any cabel
networks, oil pipelines or diodes in a transistor. Suppose there arise a situation to remove any connections that crosses the network or disturbs in transmitting signals, so that the network becomes a complete planar one, without disturbing any nodes or vertices.

References