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The planarity of bipartite graphs in R

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Abstract
Let R be a commutative ring and let Z(R) be its set of zero-divisors. We associate a graph Γ(R) to R with vertices $Z(R)^* = Z(R)-\{0\}$, the set of non- zero zero divisors of R and for distinct u, v $\in Z(R)^*$, the vertices u and v are adjacent if and only if uv = 0. In this paper, we evaluate the consistency of rectilinear crossing number of complete bipartite zero divisor graphs, in which the transformation of a non-planar graph into a planar graph is obtained by framing formula using removal of edges and removal of crossings.

Keywords: Rectilinear crossing number, planar graph, zero divisor graph

Introduction
A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a *Planar* graph, and such a drawing is called a *Planar embedding* of the graph. Let *G* be a graph drawn in the plane with the requirement that the edges are line segments, no three vertices are collinear, and no three edges may intersect in a point, unless the point is a vertex. Such a drawing is said to be a *Rectilinear drawing* of *G*. The rectilinear crossing number of *G*, denoted *cr* (*G*), is the fewest number of edge crossings attainable over all rectilinear drawings of *G* [3]. Any such a drawing is called optimal. The idea of a zero divisor graph of a commutative ring was introduced by I. Beck in [1]. The zero divisor graph is very useful to find the algebraic structures and properties of rings. We mainly focus on D. F. Anderson and P. S. Livingston's zero divisor graphs [2].

Basic Definitions
Definition - 1
If *a* and *b* are two non-zero elements of a ring Z_n such that
a.b = 0, then '*a*' and '*b*' are the *Zero divisors* of commutative
ring Z_n.

Definition - 2
If a graph $G' = (V, E')$ is a *maximum planar subgraph* of a graph $G = (V, E)$ such that there is no planar subgraph $G'' = (V, E'')$ of G with $|E''| > |E'|$, then G' is called a maximum planar subgraph of G.

The Planarity of Complete Bipartite graphs
Theorem - 1
If p and q are distinct prime numbers with $q > p$, then,
$\overline{cr}\left(\Gamma(Z_{pq})\right) = (p-1)(p-3)(q-1)(q-3)/16$ [10].

Theorem - 2
If p and q are distinct prime numbers with $q > p$, then, the consistency of Rectilinear crossing of $\overline{cr}\left(\Gamma(Z_{pq})\right)$,

(i) When removing the edges, the edge planarity is,
$P\left[E\left(\Gamma(Z_{pq})\right)\right] = pq - 3(p+q) + 9$

(ii) When removing the edges involved in crossings then, $P\left[\overline{cr}\left(\Gamma(Z_{pq})\right)\right] = n\left[E\left[\overline{cr}\left(\Gamma(Z_{pq})\right)\right]\right] - n\left[Pair\left[\overline{cr}\left(\Gamma(Z_{pq})\right)\right]\right]$

Proof:

The vertex set of $\Gamma(Z_{pq})$ is $V\left(\Gamma(Z_{pq})\right) = \{p, 2p, .. p(q-1), q, 2q, .., (p-1)q\}$. Then $\left|V\left(\Gamma(Z_{pq})\right)\right| = p + q - 2$.

By the above theorem $\Gamma(Z_{pq})$ is a bipartite graph $K_{p-1,q-1.}$ The edge set $E\left(\Gamma(Z_{pq})\right)$ is,

$$\begin{Bmatrix} (p,q), \dots (p,(p-1)q) \\ (2p,q), \dots (2p,(p-1)q) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \\ (p(q-1),q), \dots \left(p(q-1), q(p-1)\right) \end{Bmatrix}$$

Then $\left|E\left(\Gamma(Z_{pq})\right)\right| = (p-1)(q-1)$

To find the complete planarity of $\Gamma(Z_{pq})$ and thereby finding the consistency of the graph:

i. By removal of edges
ii. By removal of crossings

The complete planarity of $\Gamma(Z_{pq})$ by removal of edges

Let the complete planar graph, after the removal of edges is denoted by, $P\left[E\left(\Gamma(Z_{pq})\right)\right]$. The total number of edges is denoted by $n\left[E\left(\Gamma(Z_{pq})\right)\right] = pq - (p+q) + 1$.

The number of edges that are involved in crossings is denoted by $n\left[E_1\left(\Gamma(Z_{pq})\right)\right] = 2[(p-1) = (q-1)]$.

The number of edges that are not involved in crossings is denoted by $n\left[E_2\left(\Gamma(Z_{pq})\right)\right] = 4$ for all bipartite graphs.

Therefore we have, $P\left[E\left(\Gamma(Z_{pq})\right)\right]$

$= n\left[E\left(\Gamma(Z_{pq})\right)\right] - n\left[E_1\left(\Gamma(Z_{pq})\right)\right] - n\left[E_2\left(\Gamma(Z_{pq})\right)\right]$

$= pq - (p+q) + 1 - 2(p-1+q-1) + 4$

$= pq - 3(p+q) + 9$

The complete planarity of $\Gamma(Z_{pq})$ by removal of crossings:

Since the crossings of the edges involve pair of crossing, we proceed by defining Pair-Crossing matrix denoted by, $Pair\left[\overline{cr}\left(\Gamma(Z_{pq})\right)\right]$ and for the vertex set, V_1={$p, 2p, .., p(q-1)$} and V_2={$q, 2q, .., q(p-1)$}, we find the crossings between V_1 and V_2 denoted by $Pair\ \overline{cr}(V_1, V_2)$ and defined as,

$$\begin{bmatrix} \overline{cr}(p,q)\ \overline{cr}(2p,q)\ \dots..\overline{cr}(p(q-1),q) \\ \overline{cr}(p,2q)\ \overline{cr}(2p,2q)\ \dots..\overline{cr}(p(q-1),2q) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \overline{cr}(p,q(p-1))\dots\overline{cr}(p(q-1),q(p-1)) \end{bmatrix}$$

Summing up the crossings, $Pair\ \overline{cr}(V_1, V_2) = \sum_{m=1}^{(q-1)} \sum_{n=1}^{(p-1)} (mp, nq)$

We observe that there are $\frac{(p-1)(p-3)}{2} \sum_{n=1}^{\frac{q-3}{2}} n$ crossings.$= n[Pair\ \overline{cr}(V_1, V_2)]$

$$= n\left[pair\left(E\left(\overline{cr}\left(\Gamma(Z_{pq})\right)\right)\right)\right]$$

From theorem 1, total edge crossings is,

$$n\left[E\left(\overline{cr}\left(\Gamma(Z_{pq})\right)\right)\right] = (p-1)(p-3)(q-1)(q-3)/16$$

Removing $n\left[pair\left(E\left(\overline{cr}\left(\Gamma(Z_{pq})\right)\right)\right)\right]$, we get the complete planarity as,

$$P\left[\overline{cr}\left(\Gamma(Z_{pq})\right)\right] = n\left[E\left(\overline{cr}\left(\Gamma(Z_{pq})\right)\right)\right] - n\left[pair\left(E\left(\overline{cr}\left(\Gamma(Z_{pq})\right)\right)\right)\right]$$

$$= \left(\frac{p-1}{2}\right)\left(\frac{p-3}{2}\right)\left(\frac{q-1}{2}\right)\left(\frac{q-3}{2}\right) - \frac{(p-1)(p-3)}{2}\sum_{n=1}^{\frac{q-3}{2}} n$$

So , let us prove this by induction in p and q, $p < q$. When $p = 3, q = 5$ the graph $K_{2,q}$ is a planar graph which is trivial. So we proceed for the following cases
Case (i): Let p=5
Subcase(i): Let q=7
The vertex set of $\Gamma(Z_{pq})$ is $V(\Gamma(Z_{35})) = \{5,10,..5(q-1), q, 2q, 3q, 4q\}$. Let V_1={5,,10,..,5$(q-1)$}

and V_2=$\{q, 2q, 3q, 4q\}$. The edge set of $\Gamma(Z_{35})$ is $E(\Gamma(Z_{35})) = \begin{Bmatrix} (5,q) \dots (5,4q), (10,q) \dots (10,4q) \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ (5(q-1), q) \dots.. (5(q-1),4q) \end{Bmatrix}$

The complete planarity of $\Gamma(Z_{35})$ by removal of edges:
$n[E(\Gamma(Z_{35}))] = 24, n[E_1(\Gamma(Z_{35}))] = 20$
and $n[E_2(\Gamma(Z_{35}))] = 4$. Therefore, $P[E(\Gamma(Z_{35}))]$
$= n[E(\Gamma(Z_{35}))] - n[E_1(\Gamma(Z_{35}))]$- $n[E_2(\Gamma(Z_{35}))]$
$= 24 - 20 + 4$
=4(6)-2(4+6) +4
$= (p-1)(q-1) - 2(p-1+q-1) + 4$
$= pq - 3(p+q) + 9$

Therefore by removing 8 edges the graph $\Gamma(Z_{35})$ becomes planar.

The complete planarity of $\Gamma(Z_{35})$ by removal of crossings:

$$Pair[\overline{cr}(\Gamma(Z_{35}))] = \begin{bmatrix} 2\ 1\ 0\ 2\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \\ 2\ 1\ 0\ 2\ 1\ 0 \end{bmatrix}$$

Summing up the crossings,

$$Pair\ \overline{cr}(V_1, V_2) = \sum_{m=1}^{6}\sum_{n=1}^{4}(5m, 7n)$$

$$= Pair\ \overline{cr}\begin{Bmatrix} (5,7) + \cdots + (5,28) + (10,7) + \cdots + (10,28) \\ +(15,7) + \cdots + (15,28) + (20,7) + \cdots + (20,28) \\ +(25,7) + \cdots + (25,28) + (30,7) + \cdots + (30,28) \end{Bmatrix}$$

$$= 2[(2+0+0+2) + (1+0+0+1) + (0+0+0+0)]$$

$$= 4(2+1)$$

$$= \frac{(p-1)(p-3)}{2}\sum_{n=1}^{\frac{q-3}{2}} n$$

From theorem 1, $n\left[E\left(\overline{cr}(\Gamma(Z_{35}))\right)\right] = 12$

Therefore $P[\overline{cr}(\Gamma(Z_{35}))]$

$$= n\left[E\left(\overline{cr}(\Gamma(Z_{35}))\right)\right] - n\left[pair\ \left(E\left(\overline{cr}(\Gamma(Z_{35}))\right)\right)\right]$$

$$= 0 = 12 - 12 = (2)(1)(3)(2) - 4(2+1)$$

$$= \left(\frac{5-1}{2}\right)\left(\frac{5-3}{2}\right)\left(\frac{7-1}{2}\right)\left(\frac{7-3}{2}\right) - 4(2+1)$$

$$= \left(\frac{p-1}{2}\right)\left(\frac{p-3}{2}\right)\left(\frac{q-1}{2}\right)\left(\frac{q-3}{2}\right) - \frac{(p-1)(p-3)}{2}\sum_{n=1}^{\frac{q-3}{2}} n$$

Case (ii): Let p=7
Subcase (i): Let q=11
The vertex set of $\Gamma(Z_{pq})$ is $V(\Gamma(Z_{77})) = \{7,14,..7(q-1), q, 2q,..,6q\}$. Let V_1={7,14,..7(q − 1)} and V_2=$\{q, 2q,..,6q\}$. The edge set of $\Gamma(Z_{35})$ is

$$E(\Gamma(Z_{77})) = \begin{Bmatrix} (7,q)\ \ldots (7,6q), (14,q)\ \ldots (14,6q) \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ (7(q-1),q)\ \ldots\ldots (7(q-1),6q) \end{Bmatrix}$$

The complete planarity of $\Gamma(Z_{77})$ by removal of edges:
$n[E(\Gamma(Z_{77}))] = 60, n[E_1(\Gamma(Z_{77}))] = 32$

and $n[E_2(\Gamma(Z_{77}))] = 4$. Therefore, $P[E(\Gamma(Z_{77}))]$
$= n[E(\Gamma(Z_{77}))] - n[E_1(\Gamma(Z_{77}))]$- $n[E_2(\Gamma(Z_{77}))]$
$= 60 - 32 + 4$
=6(10)-2(6+10) +4
$= (p-1)(q-1) - 2(p-1+q-1) + 4$
$= pq - 3(p+q) + 9$

Therefore by removing 32 edges the graph $\Gamma(Z_{77})$ becomes planar.
The complete planarity of $\Gamma(Z_{77})$ by removal of crossings

$$Pair[\overline{cr}(\Gamma(Z_{77}))] = \begin{bmatrix} 0&2&4&6&8&8&6&4&2&0 \\ 0&1&2&3&4&4&3&2&1&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&1&2&3&4&4&3&2&1&0 \\ 0&2&4&6&8&8&6&4&2&0 \end{bmatrix}$$

Summing up the crossings,

$$Pair\ \overline{cr}(V_1, V_2) = \sum_{m=1}^{10}\sum_{n=1}^{6}(7m, 11n)$$

$$= Pair\ \overline{cr}\begin{Bmatrix} (7,11) + \cdots + (7,66) + \\ (14,11) + \cdots + (14,66) \\ + \cdots + \\ (77,11) + \cdots + (77,66) \end{Bmatrix}$$

$$= 2[(2+1+0+0+1+2) + (4+ +2+0+0+2+4)] + 2[(6+3+0+0+3+6) + (8+4+0+0+4+8)]$$

$$= 6(2)(1+2+3+4)$$

$$= \frac{(p-1)(p-3)}{2}\sum_{n=1}^{\frac{q-3}{2}} n$$

From theorem 1, $n\left[E\left(\overline{cr}(\Gamma(Z_{77}))\right)\right] = 120$

Therefore $P[\overline{cr}(\Gamma(Z_{77}))]$

$$= n\left[E\left(\overline{cr}(\Gamma(Z_{77}))\right)\right] - n\left[pair\left(E\left(\overline{cr}(\Gamma(Z_{77}))\right)\right)\right]$$

$$= 0 = 120 - 120 = (3)(2)(5)(4) - 6(2)(1+2+3+4)$$
$$= \left(\frac{7-1}{2}\right)\left(\frac{7-3}{2}\right)\left(\frac{11-1}{2}\right)\left(\frac{11-3}{2}\right) - 12(1+2+3+4)$$

$$= \left(\frac{p-1}{2}\right)\left(\frac{p-3}{2}\right)\left(\frac{q-1}{2}\right)\left(\frac{q-3}{2}\right) - \frac{(p-1)(p-3)}{2}\sum_{n=1}^{\frac{q-3}{2}} n$$

The Planarity of Bipartite graphs

Theorem-3:

For any graph, $\Gamma(Z_{pqr})$ where p = 2; q = 3 and r > 3 then $\overline{cr}\left(\Gamma(Z_{pqr})\right)$= cr($\Gamma(Z_{pqr})$+$(r-1)/2$.[9,10]

Theorem-4:

For any graph $\Gamma(Z_{pqr})$, p=2,q=3 and $r > 3$, the consistency of Rectilinear crossing of $\overline{cr}\left(\Gamma(Z_{pqr})\right)$,

(i) When removing the edges, the edge planarity is,

$$P\left[E\left(\Gamma(Z_{pqr})\right)\right] = \frac{pq}{2}(r-3) + (p-1)$$

(ii) When removing the edges involved in crossings then, $P\left[\overline{cr}\left(\Gamma(Z_{pqr})\right)\right]$

$$= n\left[E\left[\overline{cr}\left(\Gamma(Z_{pqr})\right)\right]\right] - n\left[Pair\left[\overline{cr}\left(\Gamma(Z_{pqr})\right)\right]\right]$$

$$= (r-1)(r-2) - [(r-1)(r-4) + (r-1) + (r-1)]$$

Proof

The vertex set of $\Gamma(Z_{pqr})$ is $V\left(\Gamma(Z_{pqr})\right) = \{p, 2p, .. q(pr-1), r, 2r, .., 5r\}$. Then $\left|V\left(\Gamma(Z_{pqr})\right)\right| = 2p(r-1) + (pq-1)$

As $\Gamma(Z_{pqr})$ is a bipartite graph [6] and can be decomposed as $K_{p-1,q-1} + K_{p-1,r-1} + K_{p-1,2(r-1)} + K_{q-1,2(r-1)} + K_{q-1,r-1}$.

The Rectilinear drawing of $\Gamma(Z_{pqr})$ follows from theorem 4.

The edge set $E\left(\Gamma(Z_{pqr})\right)$ can be obtained from the following splitted vertex sets. Let

V_1=$\{r, 2r, .., (pq-1)r\}$, V_2=$\{q, 3q, .., q(pr-1)\}$

V_3=$\{p, 2p, 4p, 7p.., q(pr-1)\}$, V_4=$\{pq, 2pq, .., (r-1)pq\}$

$$E\left(\Gamma(Z_{pqr})\right) = \begin{Bmatrix} (3r,p),(3r,2p),..(3r,p(qr-1)) \\ (3r,pq),(3r,2pq),\ldots(3r,(r-1)pq) \\ (3r,2r),(3r,4r) \\ (2r,q),(2r,3q),\ldots(2r,q(pr-1)) \\ (2r,pq),(2r,2pq),\ldots(2r,(r-1)pq) \\ (4r,pq),\ldots(4r,(r-1)pq) \\ (4r,q),\ldots(4r,q(pr-1)) \\ (r,pq),\ldots(r,(r-1)pq) \\ (5r,pq),\ldots(5r,(r-1)pq) \end{Bmatrix}$$

Then the edge set of $E\left(\Gamma(Z_{pqr})\right)$ can be splitted for convenience as follows.

$$E_1\left(\Gamma(Z_{pqr})\right) = \{(3r,4r)\}$$

$$E_2\left(\Gamma(Z_{pqr})\right) = \{(3r,p),\ldots(3r,p(qr-1))\}$$

$$E_3\left(\Gamma(Z_{pqr})\right) = \{(3r,2r)\}$$

$$E_4\left(\Gamma(Z_{pqr})\right) = \begin{Bmatrix} (3r,pq),\ldots(3r,(r-1)pq) \\ (r,pq),\ldots.(r,(r-1)pq) \\ (5r,pq),\ldots(5r,(r-1)pq) \end{Bmatrix}$$

$$E_5\left(\Gamma(Z_{pqr})\right) = E\left(\Gamma(Z_{pqr})\right) - E_1 - E_2 - E_3 - E_4$$

To find the complete planarity of $\Gamma(Z_{pqr})$ and thereby finding the consistency of the graph :

(i) By removal of edges

(ii) By removal of crossings

The complete planarity of $\Gamma(Z_{pqr})$ by removal of edges:

Let the complete planar graph, after the removal of edges is denoted by $P\left[E\left(\Gamma(Z_{pqr})\right)\right]$. The total number of edges in $\Gamma(Z_{pqr})$ is denoted by $n\left[E\left(\Gamma(Z_{pqr})\right)\right] = q^2(r-1)+p$. The number of edges that are involved in Rectilinear crossings in $\Gamma(Z_{pqr})$ are denoted by $n\left[E_3\left(\Gamma(Z_{pqr})\right)\right]$ and $n\left[E_4\left(\Gamma(Z_{pqr})\right)\right]$ and are equal to $(p-1)$ and $\frac{pq}{2}(r-3)$ respectively. The number of edges that are not involved in Rectilinear crossings are denoted and are represented by,$n\left[E_1\left(\Gamma(Z_{pqr})\right)\right] = 1 = (p-1)$,$n\left[E_2\left(\Gamma(Z_{pqr})\right)\right] = 2(r-1)$, $n\left[E_5\left(\Gamma(Z_{pqr})\right)\right]$=$q^2(r-1)+p-1-2(r-1)-1$-$\frac{pq}{2}(r-3)$. Therefore $P\left[E\left(\Gamma(Z_{pqr})\right)\right]$

$$= n\left[E_1\left(\Gamma(Z_{pqr})\right)\right] - n\left[E_2\left(\Gamma(Z_{pqr})\right)\right] - n\left[E_5\left(\Gamma(Z_{pqr})\right)\right]$$
$$= q^2(r-1)+p(p-1)-2(r-1)-q^2(r-1)+2(r-1)+\frac{pq}{2}(r-3) = \frac{pq}{2}(r-3)+1$$

Therefore removing $\frac{pq}{2}(r-3)+1$ edges from $E\left(\Gamma(Z_{pqr})\right)$, a complete planar graph is obtained.

The complete planarity of $\Gamma(Z_{pqr})$ by removal of crossings

The crossings between every two edges involve pair of crossing. So we make the calculation simple by defining Pair- crossing matrix denoted by $Pair\left[\bar{cr}\left(\Gamma(Z_{pqr})\right)\right]$. Now we find the rectilinear crossing between the vertices in the vertex sets V_1 $and\ V_2 \cup V_4 = V_1{}'$ (say). Therefore the vertices in the vertex set, will be as follows.

$$V_1{}' = \{p, 2p, ..p(qr-1), q, 2q, ..q(pr-1)\}$$

So we proceed by finding the crossings between V_1 and $V_1{}'$ denoted by $Pair\ \bar{cr}(V_1, V_1{}')$ and defined as,

$$\begin{bmatrix} \overline{(p,r)} & \dots & \overline{(p(qr-1),r)} & \overline{(q,r)} & (2q,r) & \overline{(q(pr-1),r)} \\ \overline{(p,2r)} & \dots & \overline{(p(qr-1),r)} & \overline{(q,2r)} & (2q,2r) & (q(pr-1),2r) \\ (3r,p) & \dots & (3r,p(qr-1)) & \overline{(q,3r)} & (2q,3r) & \overline{(q(pr-1),3r)} \\ \overline{(p,4r)} & \dots & \overline{(p(qr-1),4r)} & \overline{(q,4r)} & (2q,4r) & (q(pr-1),4r) \\ \overline{(p,5r)} & \dots & \overline{(p(qr-1),5r)} & \overline{(q,5r)} & (2q,5r) & \overline{(q(pr-1),5r)} \end{bmatrix} + \bar{cr}(3r,2r)$$

where the edges with bar represents that the vertices are non-adjacent . So we find the rectilinear crossings of the remaining edges. Now splitting the edge set $E_4\left(\Gamma(Z_{pqr})\right)$ for convenience as follows.

$$E_4{}'\left(\Gamma(Z_{pqr})\right) = \begin{Bmatrix} (3r,pq), \dots (3r,(r-1)pq) \\ (r,pq), \dots (r,(r-1)pq) \end{Bmatrix}$$

$$E_4{}''\left(\Gamma(Z_{pqr})\right) = \{(5r,pq), \dots (5r,(r-1)pq)\}$$

$$Pair\ \bar{cr}\left[E_4{}'\left(\Gamma(Z_{pqr})\right)\right] = (r-1)(r-4)$$

$$Pair\ \bar{cr}\left[E_4{}''\left(\Gamma(Z_{pqr})\right)\right] = (r-1)$$

$$Pair\ \bar{cr}\left[E_3\left(\Gamma(Z_{pqr})\right)\right] = (r-1)$$

Combining we get,

$$n[Pair\ \bar{cr}(V_1, V_1{}')] = (r-1)(r-4)+(r-1)+(r-1)$$
$$= n\left[Pair\ \bar{cr}\ \left(E\left(\Gamma(Z_{pqr})\right)\right)\right]$$

Since from theorem 3, total edge crossings is,

$$n\left[\ \bar{cr}\ \left(E\left(\Gamma(Z_{pqr})\right)\right)\right] = (r-1)(r-2)$$

Removing $n\left[Pair\ \bar{cr}\ \left(E\left(\Gamma(Z_{pqr})\right)\right)\right]$ we get the complete planarity as $P\left[\bar{cr}\left(\Gamma(Z_{pqr})\right)\right]$

$= n\left[E\left[\overline{cr}\left(\Gamma(Z_{pqr})\right)\right]\right] - n\left[Pair\left[\overline{cr}\left(\Gamma(Z_{pqr})\right)\right]\right]$

So let us prove this by induction on p, q and r, where $p < q < r$. Now consider for the case $p = 2, q = 3\ and\ r > q$
Case (i): Let $p = 2, q = 3$
Subcase (i): Let r = 5

The vertex set of $\Gamma(Z_{30})$ is,
$V(\Gamma(Z_{30}))$ =.{2,4,..28,3,9,..27,5,10,..25}.Let V_1={5,10,15,20,25}, V_2={3,9,21,27}, V_3={2,4,8..28}

and V_4={6,12,18,24}. The edge set of $\Gamma(Z_{30})$ is

$$E(\Gamma(Z_{30})) = \begin{Bmatrix} (5,6)\ldots(5,24),(10,3)\ldots(10,27),(10,6),..(10,24) \\ (15,2)\ldots(15,28),(15,6)\ldots(15,24),(15,10),(15,20) \\ (20,3)\ldots(20,27),(20,6)\ldots(20,24),(25,6),..(25,24) \end{Bmatrix}$$

The complete planarity of $\Gamma(Z_{30})$ by removal of edges:
$n[E(\Gamma(Z_{30}))] = 38 = q^2(r-1),$
$n[E_1(\Gamma(Z_{30}))] = \{(15,20)\} = 1 = (p-1)$
$n[E_2(\Gamma(Z_{30}))] = \{(15,2),..(15,28)\} = 8 = 2(r-1)$
$n[E_3(\Gamma(Z_{30}))] = \{(15,10)\} = 1 = (p-1)$
$n[E_4(\Gamma(Z_{30}))] = \{(5,6)..(5,24),(15,6)..(15,24),(25,6)..(25,24)\}$
$= 6 = \frac{pq}{3}(r-3)$
$n[E_5(\Gamma(Z_{30}))] = n[E(\Gamma(Z_{30}))] - n[E_1(\Gamma(Z_{30}))] - n[E_2(\Gamma(Z_{30}))] - n[E_3(\Gamma(Z_{30}))] - n[E_4(\Gamma(Z_{30}))]$
$= 22 = q^2(r-1) - 2(r-1) - \frac{pq}{2}(r-3)$

where $n[E_3(\Gamma(Z_{30}))]\ and\ n[E_4(\Gamma(Z_{30}))]$ represents number of edges that are involved in crossings and the remaining represents the number of edges that are not involved in crossings. Therefore $P[E(\Gamma(Z_{30}))]$

$= n[E(\Gamma(Z_{30}))] - n[E_1(\Gamma(Z_{30}))]$- $n[E_2(\Gamma(Z_{30}))]$- $n[E_5(\Gamma(Z_{30}))]$=38-(22+8+1)=7=$\frac{pq}{2}(r-3)$

Therefore by removing 7 edges the graph $\Gamma(Z_{30})$ becomes planar.

The complete planarity of $\Gamma(Z_{30})$ by removal of crossings
Pair cross matrix of $\overline{cr}(\Gamma(Z_{30}))$ is obtained from the rectilinear crossing between the vertex sets $V_1\ and\ V_2 \cup V_4 = V_1'$ (say).

$$Pair[\overline{cr}(V_1, V_1')] = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0 \\ 0&0&0&1&1&0&0&0 \\ 0&0&0&2&2&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix} + \overline{cr}(3r, 2r)$$

Now splitting the edge set $E_4(\Gamma(Z_{30}))$ for convenience as follows. $E_4'(\Gamma(Z_{30})) = \{(15,6)..(15,24),(5,6)..(5,24)\}$
$E_4''(\Gamma(Z_{30})) = \{(25,6)..(25,24)\}$
$Pair\ \overline{cr}[E_4'(\Gamma(Z_{30}))] = (4)(1) = (5-1)(5-4) = (r-1)(r-4)$
$Pair\ \overline{cr}[E_4''(\Gamma(Z_{30}))] = 4 = (5-1) = (r-1)$
$Pair\ \overline{cr}[E_3(\Gamma(Z_{30}))] = \overline{cr}(15,10) = 4 = (r-1)$

Combining we get, $n[Pair[\overline{cr}(V_1, V_1')]] = 12 = 4 + 4 + 4$
$= (r-1)(r-4) + (r-1) + (r-1) = n\left[Pair\left[\overline{cr}\left(E(\Gamma(Z_{30}))\right)\right]\right]$

From theorem 3 the rectilinear crossing number of $\Gamma(Z_{30})$ is
$\overline{cr}(\Gamma(Z_{30})) = 12 = (r-1)(r-2)$

Removing $n\left[Pair\left[\overline{cr}\left(E(\Gamma(Z_{30}))\right)\right]\right]$, we get the complete planarity. That is, $P[\overline{cr}(\Gamma(Z_{30}))]$
$= n\left[E[\overline{cr}(\Gamma(Z_{30}))]\right] - n\left[Pair[\overline{cr}(\Gamma(Z_{30}))]\right]$
$= 0 = 12 - 12 = 4(3) - (4 + 4 + 4)$
$(r-1)(r-2) - [(r-1)(r-4) + (r-1) + (r-1)]$

Subcase (ii): Let r = 7
The vertex set of $\Gamma(Z_{42})$ is,

$V(\Gamma(Z_{42}))$ =.{2,4,..40,3,6,9,..39,7,14,..35}.Let V_1={7,14,..,35}, V_2={3,9,15,27,33,39}, V_3={2,4,8..40} and V_4={6,12,..,36}. The edge set of $\Gamma(Z_{42})$ is

$$E(\Gamma(Z_{42})) = \left\{ \begin{array}{c} (7,6) \dots (7,36), (14,3) \dots (14,39), (14,6), .. (14,36) \\ (21,6) \dots (21,24), (21,14)(21,28) \\ (28,3) \dots (28,39), (28,6) \dots (28,36), (35,6), .. (35,36) \end{array} \right\}$$

The complete planarity of $\Gamma(Z_{30})$ by removal of edges

$n[E(\Gamma(Z_{42}))] = 56 = q^2(r-1)$
$n[E_1(\Gamma(Z_{42}))] = \{(21,28)\} = 1 = (p-1)$
$n[E_2(\Gamma(Z_{42}))] = \{(21,2), ..(21,40)\} = 12 = 2(r-1)$
$n[E_3(\Gamma(Z_{42}))] = \{(21,14)\} = 1 = (p-1)$
$n[E_4(\Gamma(Z_{42}))] = \{(21,6)..(21,42), (7,6)..(7,42), (35,6)..(35,42)\}$
$= 30 = q^2(r-1) - 2(r-1) - \frac{pq}{2}(r-3)$

where $n[E_3(\Gamma(Z_{42}))]$ *and* $n[E_4(\Gamma(Z_{42}))]$ represents number of edges that are involved in crossings and the remaining represents the number of edges that are not involved in crossings. Therefore $P[E(\Gamma(Z_{42}))]$

$= n[E(\Gamma(Z_{42}))] - n[E_1(\Gamma(Z_{42}))]$- $n[E_2(\Gamma(Z_{30}))]$- $n[E_5(\Gamma(Z_{42}))]$=56-(30+12+1)=13 =$\frac{pq}{2}(r-3)$

Therefore by removing 13 edges the graph $\Gamma(Z_{42})$ becomes planar.

The complete planarity of $\Gamma(Z_{42})$ by removal of crossings:
Pair cross matrix of $\overline{cr}(\Gamma(Z_{42}))$ is obtained from the rectilinear crossing between the vertex sets V_1 *and* $V_2 \cup V_4 = V_1{}'$ (say).

$$Pair[\overline{cr}(V_1, V_1{}')] = \begin{bmatrix} 0&0&0&3&2&0&0&2&3&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&2&2&0&0&2&2&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&2&1&0&0&1&2&0&0&0 \end{bmatrix} + \overline{cr}(3r, 2r)$$

Now splitting the edge set $E_4(\Gamma(Z_{42}))$ for convenience as follows. $E_4{}'(\Gamma(Z_{42})) = \{(21,6)..(21,42), (7,6)..(7,42)\}$
$E_4{}''(\Gamma(Z_{42})) = \{(35,6)..(35,42)\}$
$Pair\ \overline{cr}[E_4{}'(\Gamma(Z_{42}))] = 18 = (6)(3) = (7-1)(7-4) = (r-1)(r-4)$
$Pair\ \overline{cr}[E_4{}''(\Gamma(Z_{42}))] = 6 = (7-1) = (r-1)$
$Pair\ \overline{cr}[E_3(\Gamma(Z_{42}))] = \overline{cr}(21,14) = 6 = (r-1)$

Combining we get, $n[Pair[\overline{cr}(V_1, V_1{}')]] = 30 = 18 + 6 + 6$
$= (r-1)(r-4) + (r-1) + (r-1) = n\left[Pair\left[\overline{cr}\left(E(\Gamma(Z_{42}))\right)\right]\right]$

From theorem 3 the rectilinear crossing number of $\Gamma(Z_{42})$ is
$\overline{cr}(\Gamma(Z_{42})) = 30 = (r-1)(r-2)$

Removing $n\left[Pair\left[\overline{cr}\left(E(\Gamma(Z_{42}))\right)\right]\right]$, we get the complete planarity. That is, $P[\overline{cr}(\Gamma(Z_{42}))]$
$= n\left[E[\overline{cr}(\Gamma(Z_{42}))]\right] - n\left[Pair[\overline{cr}(\Gamma(Z_{42}))]\right]$
$= 0 = 30 - 30 = 4(3) - (4 + 4 + 4)$
$= (r-1)(r-2) - [(r-1)(r-4) + (r-1) + (r-1)]$

Conclusion

In this paper we find a maximum planar subgraph from a complete bipartite graphs, especially for zero divisor graphs in any rectilinear drawing of G. Note that for any zero divisor graph $\Gamma(Z_{pq})$, we get a complete bipartite graph $K_{p-1,q-1}$where p and q are primes. Therefore a complete bipartite graph is formed with even number of vertices and we suggest any bipartite or complete bipartite graph can be made into a maximum planar graph with odd number of vertices for future work. We infer from the above formulae that the removal of edges involved in crossings leading to a planar graph can be applied in any cabel networks, oil pipelines or diodes in a transistor. Suppose there arise a situation to remove any connections that crosses the network or disturbs in transmitting signals, so that the network becomes a complete planar one, without disturbing any nodes or vertices.

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