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Analysis of reverse order Law for k–EP matrices

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Abstract
In general $(AB)^\dagger \neq B^\dagger A^\dagger$, for any two matrices A and B. We say that reverse order law holds for More-Penrose inverse of the product of A and B, if $(AB)^\dagger = B^\dagger A^\dagger$ The main objective in this paper We discussed, the necessary and sufficient conditions for the reverse order law $(AB)^\dagger = B^\dagger A^\dagger$ to be hold for a pair of $_k$-EP, matrices and B.

Keywords: Matrices, penrose inverse, AXA=A

Introduction
Let C_{nxn} be the space of $n_{\times}n$ complex matrices of order n. Let C_n be the space of complex n tuples. For $A \in C_{nxn}$ let A^T, A^*, $A^\dagger$, R(A) and $\rho(A)$ denote the transpose conjugate transpose, Moore–Penrose inverse, range space, null space and rank of A respectively. A solution X of the equation AXA=A is denoted by A. A matrix $A \in C_{nxn}$ is said to be EP, if N(A) = N(A*) or R(A) = R(A*) and $\rho(A)$ = r Throughout let 'k' be fixed product of disjoint transposition in S_n= {1,2….n) and 'K' be the associated permutation matrix. A matrix $A= (\alpha_{ij}) \in C_{nxn}$ is k-hermitian if $\alpha_{ij} = {}_{\alpha\ k(j),\ k(i)}$ for j= 1,…n. A theory for k-hermitian matrices is developed in [2]. For $x = (x_{1'}, x_{2'}$ ………….. $X_n)^T \in C_n$. Let $K(X) = (X_{k(1)'} X_{k(2)'}$ ………….. $X_{k(n)})^T \in C_n$. Let $K(X) = (X_{k(1)'} X_{k(2)}$'………………….. $X\ k_{(n)}{}^T \in C_n$. A matrix $A \in C_{nxn}$ is said to be k-EP if it satisfies the condition $Ax = 0 \Leftrightarrow A^*K(X)=0$ (or) equivalently N(A) N(A*K) (or) R(A) = R(KA*) Moreover, A is said to be k-EP, if A is k-EP and $\rho(A)$ = r further properties of k-EP matrices one may refer [3]. In [4] it is shown that if A, B and AB are Ep_r matrices then AB, BA $A^\dagger B^\dagger$ are all k-EP, matrices.
In this paper, necessary and sufficient condition for the reverse order law $(AB)^\dagger = B^\dagger A^\dagger$ to be hold for an k-EP, matrices A and B are discussed.

Reverse Order Law for-EP_r Matrices
For any two singular matrices A, $B \propto C_{nxn}$ $(AB)^{-1} = B^{-1} A^{-1}$ holds. However, it is not true generalized inverse of matrices. In general $(AB)^\dagger \neq B^\dagger A^\dagger$ for any two matrices A and B. For example,

$$A = [1, 0],\ B = \begin{pmatrix}1\\1\end{pmatrix}, \qquad AB = [1]$$

$(AB)^\dagger$ = [1]. $B^\dagger A^\dagger \neq (AB)^\dagger$. It is well known that (p. 181, [I]), $(AB)^\dagger = B^\dagger A^\dagger$ if any only if, $R(BB^* A^*) \subseteq R(A^*)$ and $R(A^*AB) \subseteq R(B)$.

Theorem 2.1: If A, B are k-EP_r Matrices with R(A) = R(B*) then $(AB)^\dagger = B^\dagger A^\dagger$.
Proof: Since A is k-$EP_{r'}$ R(A) =R(KA*)
$\Rightarrow$ R(B*) = R(KA*) (By hypothesis)
$\Rightarrow$ R(KB) = R(KA*) (Since Bis k-EP)
$\Rightarrow$ R(B) = R(A*) (Since R (KA) = R(KB)
$\Rightarrow$ R(A) =R(B))
$\Rightarrow$ $R(B) = R(A^\dagger)$ (By [1], $R(A^*) = R(A^\dagger)$)

Now, in order to prove $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$, it is enough if we prove that $B^{\dagger}A^{\dagger}$ satisfies the difining four equation of the Moore-Penrose inverse, that is

$(AB)(B^{\dagger}A^{\dagger})AB = AB \rightarrow (1)$
$(B^{\dagger}A^{\dagger})(AB)(B^{\dagger}A^{\dagger}) = (B^{\dagger}A^{\dagger}) \rightarrow (2)$
$[(AB)(B^{\dagger}A^{\dagger})]^* = (AB)(B^{\dagger}A^{\dagger})] \rightarrow \ (3)$
$[(B^{\dagger}A^{\dagger})(AB)]^* = (B^{\dagger}A^{\dagger})(AB)] \rightarrow \ (4)$
Now, by [1], $R(A) = (RB) \Leftrightarrow AA^{\dagger} = BB^{\dagger}$

$\therefore R(A^{\dagger}) = R(B) \Rightarrow A^{\dagger}(A^{\dagger})^{\dagger} = BB^{\dagger}$
$\Rightarrow A^{\dagger}A = BB^{\dagger}$

Hence, $(AB)(B^{\dagger}A^{\dagger})(AB) = ABB^{\dagger}(A^{\dagger}A)B$

$= AB(B^{\dagger}BB^{\dagger})B$
$= A(BB^{\dagger}B)B$
$= AB$
$(AB)(B^{\dagger}A^{\dagger})(AB) = AB$

Thus (1) is proved
Now, $(B^{\dagger}A^{\dagger})(AB)(B^{\dagger}A^{\dagger}) = B^{\dagger}(A^{\dagger}A)BB^{\dagger}A^{\dagger}$
$= (B^{\dagger}BB^{\dagger})BB^{\dagger}A^{\dagger}$
$= (B^{\dagger}BB^{\dagger})A^{\dagger}$
$= B^{\dagger}A^{\dagger}$
$(B^{\dagger}A^{\dagger})(AB)(B^{\dagger}A^{\dagger}) = B^{\dagger}A^{\dagger}$

Thus (2) is proved
Now, $R(B) = R(A^{\dagger}) \Rightarrow$ given $x \in C_n$ there exist a $y \in C_n$ such that $A^{\dagger}y = Bx$.

Therefore,
$A^{\dagger}y = Bx \Rightarrow (ABB^{\dagger})A^{\dagger}y = (ABB^{\dagger})Bx$
$\Rightarrow (AB)(B^{\dagger}A^{\dagger})y = A(BB^{\dagger}B)X$
$\Rightarrow (AB)(B^{\dagger}A^{\dagger})y = A(Bx)$
$\Rightarrow (AB)(B^{\dagger}A^{\dagger})y = (AA^{\dagger})y$

Since $AA^{\dagger}$ is hermitian, it follow that $(AB)(B^{\dagger}A^{\dagger})$ is Hermitian.

i.e. $[(AB)(B^{\dagger}A^{\dagger})]^* = [(AB)(B^{\dagger}A^{\dagger})]$

Thus (3) is proved.
Now,
$R(B) = R(A^{\dagger}) \Rightarrow$ given $X \in C_n$ there exist a $y \in C_n$ such that $Bx = A^{\dagger}y$.
Therefore, $Bx = A^{\dagger}y \Rightarrow (B^{\dagger}A^{\dagger}A)Bx = (B^{\dagger}A^{\dagger}A)A^{\dagger}y$.
$\Rightarrow (B^{\dagger}A^{\dagger})ABx = B^{\dagger}(A^{\dagger}AA^{\dagger})y$.
$\Rightarrow (B^{\dagger}A^{\dagger})(AB)x = B^{\dagger}A^{\dagger}y$.
$\Rightarrow (B^{\dagger}A^{\dagger})(AB)x = B^{\dagger}B(X)$.

Since $(B^{\dagger}B)$ is hermitian, it follow that $(B^{\dagger}A^{\dagger})(AB)$ is Hermitian.

(i.e.) $[(B^{\dagger}A^{\dagger})(AB)]^* = [B^{\dagger}A^{\dagger})(AB)]$

Thus (4) proved. Thus $B^{\dagger}A^{\dagger}$ satisfies all the four equation of the Moore-penrose inverse Hence the theorem.

Remark 2.2: In the above Theorem the condition that R (A) = R (B*) is essential.

Example 2.3

Let $= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

and $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ A and B are k EP, matrices

$AB = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \rho(AB) = 1$

$R(A) \neq R(B^*), A^{\dagger} = 1/14 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B^{\dagger} = \begin{pmatrix} 0 & 1 \\ 0 & C \end{pmatrix}$

$B^{\dagger}A^{\dagger} = 1/14 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, (AB)^{\dagger} = 1/12 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

Thus $(AB)^{\dagger} \neq B^{\dagger}A^{\dagger}$

Remark 2.4
The converse of Theorem 2.1 need not be true in general.

For, let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

and $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ A and Bae k- EP_1 matrices.

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A^{\dagger} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$B^{\dagger} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $(AB)^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

$B^{\dagger}A^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$

But, $R(A) \neq R(B^*)$

Note 2.5: The validity of the converse of the Theorem 2.1 is proved under certain conditions.

Corollary 2.6: If A, B are k-EP_r matrices with $\rho(AB) = r$ and $(AB)^{\dagger} = (AB)^{\dagger} = B^{\dagger}A^{\dagger}$, then $R(A) = R(B^*)$

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