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An empirical comparison of growth models

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Abstract
The aim of this paper is to compare the exponential growth model and the logistic growth model. Although these models may seem similar in terms of their formulas, graphically, the exponential growth model lacks realism, while the logistic growth model offers a more practical approach. Specifically, the exponential growth model assumes no carrying capacity for the Earth, which contradicts reality since our planet has limited natural resources. In contrast, the logistic growth model addresses this limitation and offers a more accurate representation of population growth. Therefore, in this paper, we will provide a detailed analysis of the differences between these two models and highlight the advantages of the logistic growth model over the exponential growth model.

Keywords: Carrying capacity, exponential growth model, more realistic form of exponential growth model, logistic growth model, population growth

Introduction

The rapid growth of the global population is a critical concern for sustainable development. In order to understand and address this issue, mathematical models have been developed to help explain the dynamics of population growth. The exponential growth model is one of the most basic models used to describe the growth of a population, but it has several limitations. For example, it assumes that population growth will continue at a constant rate indefinitely, which is not a realistic assumption. It was first introduced by Thomas Robert Malthus (English Economist) in 1798 who had proposed the idea of exponential growth model also known as J shaped Growth Curve. In contrast, the logistic growth model is a mathematical equation that describes the growth rate of a population or system proportional to its size, taking into account limiting factors that eventually slow down growth. It was first introduced by the Belgian mathematician Pierre François Verhulst in 1838 to model population growth, and has since been widely adopted in many fields. This model is also known as S shaped growth curve. The logistic growth model has been used in many studies to predict future trends of populations or systems. For example, a study by Liu et al. (2015) used the logistic growth model to predict the future growth of the human population and its impact on natural resources. The study estimated that the world population would reach 9.6 billion by 2050 and 10.9 billion by 2100, and that the demand for food, water, and energy would increase accordingly. The study emphasized the importance of sustainable development and resource management to ensure the long-term survival of the human population. Another study by Nyström et al. (2012) used the logistic growth model to predict the future trends of fish populations in the Baltic Sea. The study estimated the intrinsic growth rate and carrying capacity of several fish species based on historical data, and used the model to predict future catch levels under different management scenarios. The study concluded that sustainable management practices could increase the long-term yield of fish populations, while overfishing could lead to their collapse. The model assumes that the growth rate of a population is proportional to the product of the current population size and the difference between the carrying capacity and the current population size. The carrying capacity is the maximum population size that the environment can support, and represents a limit to growth.

This paper aims to critically review the mathematical models used to describe population growth, with a specific focus on the exponential growth model and the logistic growth model. We will explore the key features of each model, their strengths and weaknesses, and their applications in different contexts. By examining these models in detail, we hope to provide insights into the dynamics of population growth and help inform policy decisions that can support sustainable development. The paper is organized as follows. Section II provides an overview of the study on the both growth model and its application in forecasting future trends. Section III describes the Formulation and methodology used in the study, including data collection and parameter estimation techniques for both the models. Section IV discusses the implications of the study for decision-making in various fields and highlights the strengths and limitations of the logistic growth model for predictive analysis. Finally, Section V concludes the paper with a summary of the main findings and suggestions for future research.

Overview of the Study: Exponential Growth Model basically tells us that there are unlimited natural resources which clearly states that as time keep on increasing the population will also keeps on increasing and according to this model there will not be any kind of decline in growth rate as number of individuals taken birth will always be greater than the mortality rate of individuals while Logistic Growth Model is actually a generalization of exponential growth model where it shows that earth has carrying capacity and population cannot exceed this limit, when population growth reaches to its carrying capacity in that case no further growth of population can occur which states that at that stage growth rate will be approximately zero as number of individuals birth and death will be approximately same.

Formulation of Exponential Growth Model: According to definition of exponential growth model, we conclude that population grows at a rate directly proportional to current size of population i.e., rate of change of population with respect to time is directly proportional to the current population size.

i.e., $dM/dt = AM$ -----(1)

Where M is the original population size
A is a constant and is equal to difference between birth rate and death rate i.e. $A= b-d$
Where b = number of individuals taken birth and d = mortality number. Using variable seperation we have

$dM/M= A\, dt$ ------(2)

Now Integrating (2), we get

$M = e^{At + C}$ -------(3)

Now when time t= 0 obviously there would be some population, let say M_o

Then using this condition in (3) we get: -

$M_o = e^C = C$ ---- (4)

Putting (4) we get: -

$M= M_o \times e^{At}$ ----- (5)

Which is a required exponential growth model.

Assumption and Limitation

There were few assumptions of exponential growth model

- One of important assumption made in this model is constant growth rate as we have already discussed that A is constant which a growth rate irrespective of the factors that can affect it.
- There should be no immigration and emigration of the population.
- Another assumption that is somewhere interrelated with above assumption is there should be no genetic variation among the individuals.
- Variation of ages and sizes of population members were also not included in exponential growth model.

Limitation of exponential growth model: One of the major limitation of exponential growth model is " if you consider population of particular area grows 2% per year it looks very small for short term but if you observe it carefully this will be doubled in almost 35 - 37 years and get 4 times after 70 – 74 years as there is not any kind of carrying capacity of our planet in exponential growth model, So in this model having unlimited resources it cannot get finished up even after 70 years." But in reality, the number of resources we have is not unlimited and exponential growth model have any variable which indicate maximum limit or kind of limited resources we have or any other factors that can affect population growth rate.

Methodology of exponential growth model: The methodology used in this study involved the application of the exponential growth model to historical data to forecast future trends of a population or system. It generally seen in bacteria as they are prokaryotic i.e., one bacteria divides into two, two into four and so on so far. We use simulated data and this simulated data represents the population size of a Bacteria growth species over a time

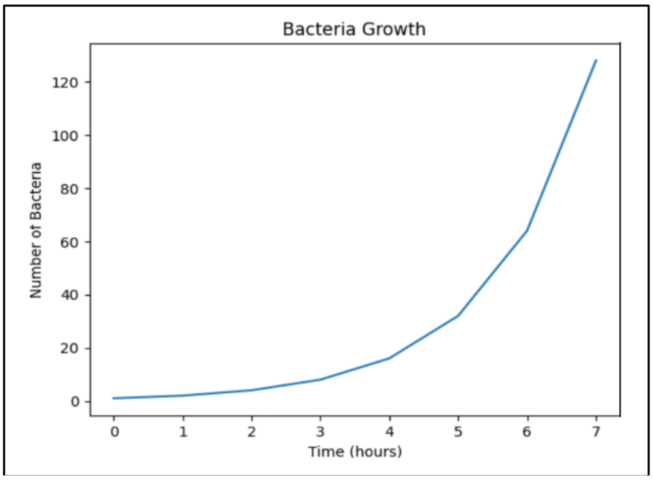

Fig 1: Bacteria Growth over time

In Figure 1, Bacteria growth graph shows the number of bacteria over time. The x-axis represents time in hours, while the y-axis represents the number of bacteria. The graph shows exponential growth, where the number of bacteria doubles every hour. This is typical of exponential growth models, where the growth rate is proportional to the size of the population. The graph starts with just one bacterium and grows rapidly over time. By the end of the 7th hour, there are over 100 bacteria. The steepness of the curve represents the rate of growth, with the curve becoming increasingly steep as the number of bacteria increases.

Time (Hours)	Bacteria Growth
0	1
1	2
2	4
3	8
4	16
5	32

Criticism of exponential growth model: As Exponential Growth Model fully based on ideal condition and do not consider any external factor that can affect the population growth rate whether it is natural resources, shelter, adequate nutrition or disease that can easily transmit from person to person due to large population. It will keep on increasing with positive constant growth rate. This is one of the reason for criticism of exponential growth model. We can consider exponential growth model for short term as in this period obviously we have unlimited number of natural resources but for long term it is not well and good model for future prediction of population growth.

Formulation of Logistic Growth Model

To have original logistic growth model we will consider Taylor's theorem for its derivation

Consider Dx/dt = F(M) i.e., rate of growth is a function of original population size Then, expanding F(M) using Taylor's series

$F(M) = F(0) + b F'(0) + b F''(0) +$ ----------

Then putting value of F(M) we have

$dM/dt = A_1 + A_2 M + A_3 M^2 +$ ------------

On applying condition that at t = 0, obviously population will be zero and secondly on reaching maximum limit dM/dt = 0 at $M = M_{max}$

Then we have $dM/dt = A (1 - M/K)$ -----(2)

On integrating (2) we have

$M(t) = KM_o / (K - M_0) \times e^{-At} + M_o$

Which is a required logistic growth model.

Assumption and Limitation of Logistic Growth Model

The major feature of this logistic growth model is

(1) The term $1/N \times dM/dt$ (relative growth rate) becomes smaller and smaller with increasing population size.

(2) Population will ultimately become equal to K (carrying capacity) when taking limit 't' tending to infinity which implies after millions of years the population on earth will be equal to its carrying capacity.

(3) Population at a inflexion point is exactly half of the carrying capacity, i.e. $M_{inf} = K/2$.

Methodology of Logistic Growth Model

The methodology used in this study involved the application of the logistic growth model to historical data to forecast future trends of a population or system. We use simulated data and this simulated data represents the population size of a particular wildlife species over a period of 11 years, from 2010 to 2020. The data in figure 2 shows a gradual increase in population size over time, with some fluctuations from year to year. This data can be used as historical data in a predictive analysis using the logistic growth model to forecast future population sizes of the wildlife species. The data can also be used to estimate the model parameters, such as the carrying capacity and growth rate, which are necessary for making accurate forecasts.

Year	Population size
2010	100
2011	110
2012	115
2013	130
2014	135
2015	142
2016	148
2017	155
2018	160
2019	165
2020	170

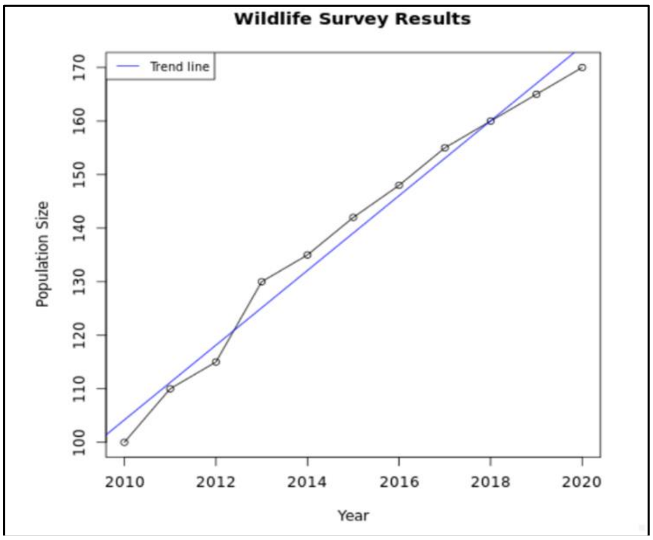

Fig 2: Wildlife species over a period from 2010 to 2020

The logistic growth model is a valuable tool for predictive analysis in various fields, but its application requires careful consideration of the assumptions made and the validity of the parameter estimates. This study highlights the importance of accurate parameter estimation and the need to validate models with new data.

Application of Logistic Growth Model

Logistic Growth Model have many applications in various fields, we will discuss only two applications of this model

(1) For Production Forecasting: This model can be used to forecast production in oil or gas wells. We have learnt that anything will low permeability declines hyperbolically (in this case it is oil or gas), so basically generalized logistic growth model which is a generalized form of classical logistic growth model which can accommodate numerous curves is used to forecast the production in oil or gas wells.

(2) For telecommunication: In business, to invest on emerging technologies or on any planning strategies will obviously focus on how much risk is there in investing on that technology or strategy, will it be safe for anyone to invest on any type of technology without have its full knowledge, these are basically some questions arises in everyone's mind but for this there is method of Forecasting that gives answer of such questions.

Similarly, to introduce a new product in market a basic problem arises is forecasting the rate of market development over time. There are number of mathematical models which represent the time pattern of diffusion process i.e., the time by which a product that is to be introduced is adopted by everyone. The model that is quite relevant to this type of problem is logistic growth model because initially when a product is introduced it is adopted by only number but after sometime knowing it's efficiency it's demands increases and show a S shaped Growth curve which is logistic growth model (i.e. there will be a point which it's growth will be very less such that it is seen touches the upper bound because that product will not be only one of such kind there will be another product that will be introduced over time with more better feature than previous one which ultimately decline it's growth rate.

Table 1: Similarities and Differences of Logistic and Exponential Growth Model

Exponential Growth Model	Logistic Growth Model
This growth has J shaped Growth curve.	This growth curve has S shaped or sigmoid growth curve.
It depends only on original size of population or simply we can say on size of population.	It not only depends on size of population but also on number of resources present in the respective environment.
It does not reach to stationary phase frequently.	It reaches to its stationary phase.
Do not have any kind of upper limit or carrying capacity.	It has an upper limit or we can say a carrying capacity imposed by the respective environment.
It can be applied to those species only which do not have carrying capacity factor or can be applied for short term period. For example: - in case of amoeba, prokaryotic, etc.	It can be applied to any kind of population which has upper limit (carrying capacity) to grow.
It has two phase lag phase and log phase only.	It has four phase lag phase, log phase, deceleration phase and stationary phase.
Population can crash due to mass mortality that could be due to factors like earthquake, tsunami or any other natural disaster.	Population can crash there this any kind of boundation that it cannot crash but generally it does not happen.

Advantages of Exponential Growth Model and Logistic Growth Model

In Exponential Growth Model :- This model without carrying capacity is reliable for the population to grow up without any boundation and since in this growth resources are limited, so there will not be any kind of competition amongst any individual of getting desirable amount of resources.

In Logistic Growth Model: - This model with carrying capacity always have a checks on increasing population and when it reaches nearer to carrying capacity growth will be in stationary phase.

Disadvantage of Exponential Growth Model and Logistic Growth Model

In Exponential Growth Model: As discussed in its advantage the population will grow up without any checks but as resources starts exhausting this will lead to crash situation and due to which there will a mass mortality as the population increased so much such that it will definitely crashes in case of exhausted natural resources or low resources and ultimately will increase competition amongst individuals which definitely leads to mass mortality.

In Logistic Growth Model: As population keeps on increasing upto carrying capacity with yet limited amount of resources will leads to competition amongst individuals in logistic growth model also, but in this case there will not be any mass mortality. There will be only type of checks and reduce to make the population under maximum limit.

Conclusion

The brief conclusion of the Study is basically a comparison two classical growth model i.e. exponential and logistic growth model where exponential is applicable only if there is no boundation on the growth rate of population and also can be applicable for short term of period whereas logistic growth model as it's name suggests i.e. logistic means logical though its not correct meaning of logistic but yet it more precisely calculates the growth rate of the population since in reality we do not have unlimited resources and logistic growth model includes that point as well in form of maximum limit which were excluded from exponential growth model and this maximum limit not only fix the growth rate but also do not allow population growth rate to increase further if it reaches to carrying capacity, if so then there will be a crash like situation to make the population within carrying capacity.

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