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Computation of square root of a number using centroidal mean and its invariant

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Abstract
Abstract. In this article the invariant centroidal mean is introduced. A pair of double sequences in term of Centroidal mean and its invariant are defined discussed the properties monotonicity log-convexity and log-concavity. Finally as an illustration it is justified that the new Gaussian compound mean $CT' \otimes CT$ converging faster than Gaussian compound mean $H \otimes A$. 2000 *Mathematics Subject Classification.* Primary 26D15.

Keywords: Arithmetic mean geometric mean harmonic mean centroidal mean and invariant centroidal mean

1. Introduction

For $a\ b > 0$ the well-known means in literature are $A(a\ b) = \frac{a+b}{2}$ $G(a\ b) = \sqrt{ab}$ $H(a\ b) = \frac{2ab}{a+b}$ and $C(a\ b) = \frac{2}{3}\left(\frac{a^2+ab+b^2}{a+b}\right)$ are respectively called Arithmetic Geometric Harmonic and Centroidal mean. The various interesting results are found in [3, 5] and Researchers discussed about double sequences.

As an illustration [6] the popular iteration method called Heron's iteration method is used to extract the square root of any positive number from Gaussian double sequences given by $a_{n+1} = H(a_n\ b_n)$and $b_{n+1} = A(a_n\ b_n)$also used the Archimedean double sequences $a_{n+1} = A(a_n\ b_n)$ and $b_{n+1} = G(a_{n+1}\ b_n)$ to get the approximate value of π. In [1 6] Nagaraja *et al*. were discussed logarithmic convexity and logarithmic concavity of Archimedean and Gaussian double sequences. This work motivated to develop this article.

2. Definitions and Results

Definition 2.1. [6] A mean is put forth as a function $f: R_+^2 \to R_+$ which has the property where $r \wedge s = min(r\ s)\ and\ r \vee s = max(r\ s)$

Definition 2.2. [6] A mean N is P-complementary to M if it satisfies $P(MN) = P$. Suppose a given mean M has a unique G-corresponding mean N is denoted by

$$N = M^{(G)} = \frac{G^2}{M}$$

Then the invarient centroidal mean is defined as $CT' = \frac{3}{2}\left(\frac{ab(a+b)}{a^2+ab+b^2}\right)$

Definition 2.3. [6] The double sequences in terms of invariant centroidal mean and cen-troidal means are defined as;

$$a_{n+1} = CT'(a_n\ b_n) = \frac{3}{2}\left(\frac{ab(a+b)}{a^2+ab+b^2}\right) \text{ and } b_{n+1} = CT(a\ b) = \frac{2}{3}\left(\frac{a^2+ab+b^2}{a+b}\right)$$

Definition 2.4. [1] The sequence c_n is said to be log-convex if $c_n^2 \leq c_{n+1}c_{n-1}$ otherwise it is called log-concave.
Definition 2.5. [6] The sequence $(r_n)_n \geq 0$ *and* $(s_n)_n \geq 0$ *given* $r_{n+1} = M(r_n\ s_n)\ s_{n+1} = N(r_n\ s_n)\ n \geq 0$ *is* called a Gaussian double sequence.

Lemma 2.1. The Invariant Centroidal mean for two distinct positive real values a and b isa mean.
Proof: The proof of this lemma is discussed by considering two cases as below:

Case (i): For $a < b$

Consider $a - CT'(a\ b) = a - \frac{3}{2}\left(\frac{ab(a+b)}{a^2+ab+b^2}\right) = \left(\frac{(a-b)(2a+b)}{2(a^2+ab+b^2)}\right) < 0$

Which gives $a - CT'(a\ b) < 0$

$a < CT'(a\ b) = 0$ and hence $CT'(a\ b) > a$

Case (II): For $a < b$

Consider $CT'(a\ b) - b = \frac{3}{2}\left(\frac{ab(a+b)}{a^2+ab+b^2}\right) - b = \frac{b}{2}\left(\frac{(a-b)(2a+b)}{(a^2+ab+b^2)}\right) < 0$

Which gives $CT'(a\ b) - b < 0$ and hence $CT'(a\ b) < b$

Combining both the above cases $Min(a\ b) < CT'(a\ b) < Max(a\ b)$.

Therefore $a < CT'(a\ b) < b$ Satisfy the condition to be a mean

Hence the proof of lemma 2.1

Property: Since $CT'(a\ b) = CT'(a\ b)$ and $CT'(ta\ tb) = tCT'(a\ b)$ it is clear that invariant to centroidal mean is symmetric and homogeneous respectively.

Theorem 2.1. For two distinct positive real values $a_n < b_n$ *the sequence* $a_{n+1} = CT'(a_n\ b_n)$is monotonically increasing and the sequence $b_{n+1} = CT(a_n\ b_n)$ is monotonically decreasing. Also satisfy

$min(a\ b) = a = a_0 < a_1 < a_2 <..< a_n < a_{n+1} <$

$... < b_{n+1} < b_n <..< b_1 < b_0 = b = max(a\ b)$

Proof: Let $a_{n+1} = CT'(a_n\ b_n) = \frac{3}{2}\left(\frac{ab(a+b)}{a^2+ab+b^2}\right)$ and $b_{n+1} = CT(a\ b) = \frac{2}{3}\left(\frac{a^2+ab+b^2}{a+b}\right)$

$$\frac{a_{n+1}}{a_n} = \frac{3b_n(a_n + b_n)}{2(a_n^2 + a_nb_n + b_n^2)} > 1$$

Gives $a_{n+1} > a_n$ which hold for all n

This proves that
(2.1) $Min(a\ b) = a = a_0 < a_1 < a_2 <....< a_n < a_{n+1}$

Similarly

$$\frac{b_{n+1}}{b_n} = \frac{2(a_n^2 + a_nb_n + b_n^2)}{3b_n(a_n + b_n)} < 1$$

Gives $b_{n+1} > b_n$ which hold for all n

This proves that

(2.2) $b_{n+1} < b_n <....< b_1 < b_0 = b = \max(a\ b)$

Eqs (2.1) and (2.2) leads to theorem 2.1

Theorem 2.2. For $n \geq 0\ a_n < b_n$ *the sequence*$a_{n+1} = CT'(a_n\ b_n)$ *is log-concave and thesequence* $b_{n+1} =$

$CT(a_n\ b_n)$ *is log-convex.*

Proof: If $a_n < b_n$ the Centroidal mean and invariant to Centroidal mean are given by

$b_{n+1} = CT(a_n\ b_n) = \frac{2}{3}\frac{(a_n^2+a_nb_n+b_n^2)}{(a_n+b_n)}$ and $a_{n+1} = CT'(a_n\ b_n) = \frac{3}{2}\left(\frac{a_nb_n(a_n+b_n)}{a_n^2+a_nb_n+b_n^2}\right)$

$$\frac{a_n}{a_{n+1}} - \frac{a_{n-1}}{a_n} = \frac{2(a_n^2+a_nb_n+b_n^2)}{3b_n(a_n+b_n)} - \frac{2(a_{n-1}^2+a_{n-1}b_{n-1}+b_{n-1}^2)}{3b_{n-1}(a_{n-1}+b_{n-1})}$$

$$= \frac{2(a_n^2a_{n-1}b_{n-1} + a_n^2b_{n-1}^2 - a_nb_na_{n-1}^2 - a_{n-1}^2b_n^2)}{3b_nb_{n-1}(a_{n-1}+b_{n-1})(a_n+b_n)}$$

$$= \frac{2(a_na_{n-1} + a_nb_{n-1} + a_{n-1}b_n)(a_nb_{n-1} - b_na_{n-1})}{3b_nb_{n-1}(a_{n-1}+b_{n-1})(a_n+b_n)}$$

Since $a_0 < a_1 < a_2 < \ldots. < a_n < b_n < b_{n-1} < \ldots\ldots\ldots < b_1 < b_0$

$a_nb_{n-1} > b_na_{n-1}$

$a_nb_{n-1} - b_na_{n-1} > 0$

$(since\ a_{n-1} < a_n\ and\ b_n < b_{n-1})$

$\frac{a_n}{a_{n+1}} - \frac{a_{n-1}}{a_n} > 0$

So $a_n^2 > a_{n+1}a_{n-1}$ and hence $a_{n+1} = CT'(a_n\ b_n)$ is log-concave

Similarly Consider

$$\frac{b_n}{b_{n+1}} - \frac{b_{n-1}}{b_n} = \frac{3b_n(a_n+b_n)}{2(a_n^2+a_nb_n+b_n^2)} - \frac{3b_{n-1}(a_{n-1}+b_{n-1})}{2(a_{n-1}^2+a_{n-1}b_{n-1}+b_{n-1}^2)}$$

$$= \frac{3(a_nb_na_{n-1}^2 + b_n^2a_{n-1}^2 - a_n^2a_{n-1}b_{n-1} - b_{n-1}^2a_n^2)}{2(a_n^2+a_nb_n+b_n^2)(a_{n-1}^2+a_{n-1}b_{n-1}+b_{n-1}^2)}$$

$$= \frac{3(b_na_{n-1} - a_nb_{n-1})(a_na_{n-1}+b_na_{n-1} - b_{n-1}a_n)}{2(a_n^2+a_nb_n+b_n^2)(a_{n-1}^2+a_{n-1}b_{n-1}+b_{n-1}^2)}$$

Since $a_0 < a_1 < a_2 < \ldots. < a_n < b_n < b_{n-1} < \ldots\ldots\ldots < b_1 < b_0$

$a_nb_{n-1} > b_na_{n-1}$

$b_na_{n-1} - a_nb_{n-1} < 0$

$(since\ a_{n-1} < a_n\ and\ b_n < b_{n-1})$

$\frac{b_n}{b_{n+1}} - \frac{b_{n-1}}{b_n} < 0$

So $b_n^2 < b_{n+1}b_{n-1}$ and hence$b_{n+1} = CT(a_n\ b_n)$ is log –convex. Thus the Proof of theorem 2.2 completes.

Theorem 2.3. *The sequences* $(a_n)_n \geq 0$ *and* $(b_n)_n \geq 0$ *are defined in terms of invariant to centroidal mean and centroidal mean which convergent to the common limit depicted as* $CT(a\ b) \otimes CT'(a\ b) = G(a\ b) = \sqrt{x}$.

Proof: We know that $a_n < a_{n+1} < b_{n+1} < b_n\ n \geq 0$ (2.3)

$b_{n+1} - a_{n+1} = \frac{(4a_n^4+4b_n^4-a_n^3b_n-a_nb_n^3-6a_n^2b_n^2)}{6(a_n^3+b_n^3+2a_n^2b_n+2a_nb_n^2)}$

Consider $\frac{(4a_n^4+4b_n^4-a_n^3b_n-a_nb_n^3-6a_n^2b_n^2)}{6(a_n^3+b_n^3+2a_n^2b_n+2a_nb_n^2)} - (b_n - a_n)$

(2.4) $\frac{(a_n-b_n)(14a_n^3+6b_n^3+19a_n^2b_n+13a_nb_n^2)}{6(a_n^3+b_n^3+2a_n^2b_n+2a_nb_n^2)} < \frac{(b_n-a_n)}{6}$

Combining the eqs (2.3) and (2.4) gives

$$b_{n+1} - a_{n+1} < \frac{b_n - a_n}{6}$$

Repeat the process leads to

$$b_{n+1} - a_{n+1} < \frac{b_n - a_n}{6} < \frac{b_{n-1} - a_{n-1}}{6} < \cdots < \frac{b-a}{6}$$

So as $n \to \infty$ $b_{n+1} - a_{n+1} \to 0$ and

$$\lim_{n\to\infty} b_{n+1}a_{n+1} = \lim_{n\to\infty} b_n a_n = \ \dots\dots = ab = \sqrt{ab}\sqrt{ab}$$

Take $a = 1$ $b = x$ leads to

$$\lim_{n\to\infty} b_{n+1}a_{n+1} = \sqrt{x}\sqrt{x} = x$$

Thus by Theorem 2.1 $(a_n)_n \geq 0$ *and* $(b_n)_n \geq 0$ are monotonically increasing and monoton- ically decreasing sequences respectively. Also

$$a_{n+1}b_{n+1} = CT'(a_n\ b_n)CT(a_n\ b_n) = \left(\frac{a_nb_n(a_n+b_n)}{a_n^2+a_nb_n+b_n^2}\right)\left(\frac{(a_n^2+a_nb_n+b_n^2)}{(a_n+b_n)}\right) = a_nb_n$$

Where x is a multiple of two positive real numbers. This implies that
(2.5)

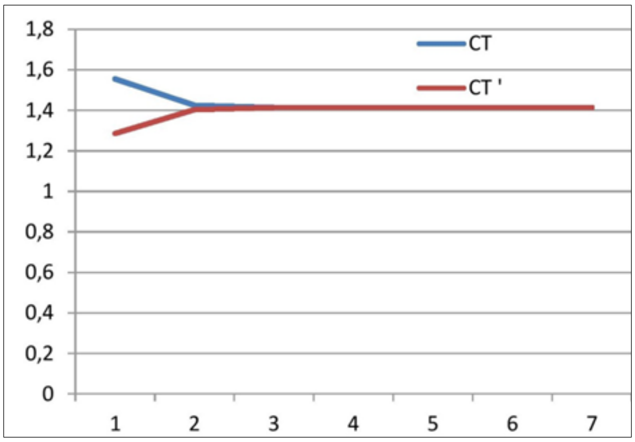
$$\lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_n = \ x$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \ \sqrt{x}$$

Therefore the sequence "$(a_n)_n \geq 0$ *and* $(b_n)_n \geq 0$" are convergent to a common limit $\sqrt{x}$.
Hence the proof of theorem 2.3 completes.

3. Application of extracting square root

In [6] authors discussed Heron's method of extracting square roots using Gaussian compound mean $H \otimes A$. In this section the convergence process of the new Gaussian compound mean $CT' \otimes CT$ and $H \otimes A$ are discussed.
The following table-1 and figures (1) and (2) illustrate the approximate process of computing $\sqrt{2}$. Also evident that $CT' \otimes CT$ is convergence to common limit faster than $H \otimes A$.



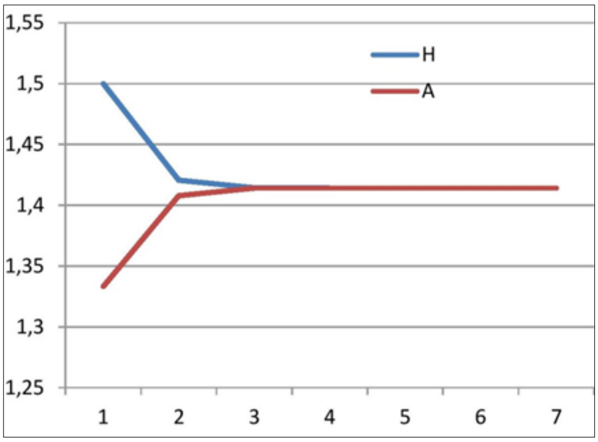


Fig 1: Graphs of Gaussian and centroidal compound means

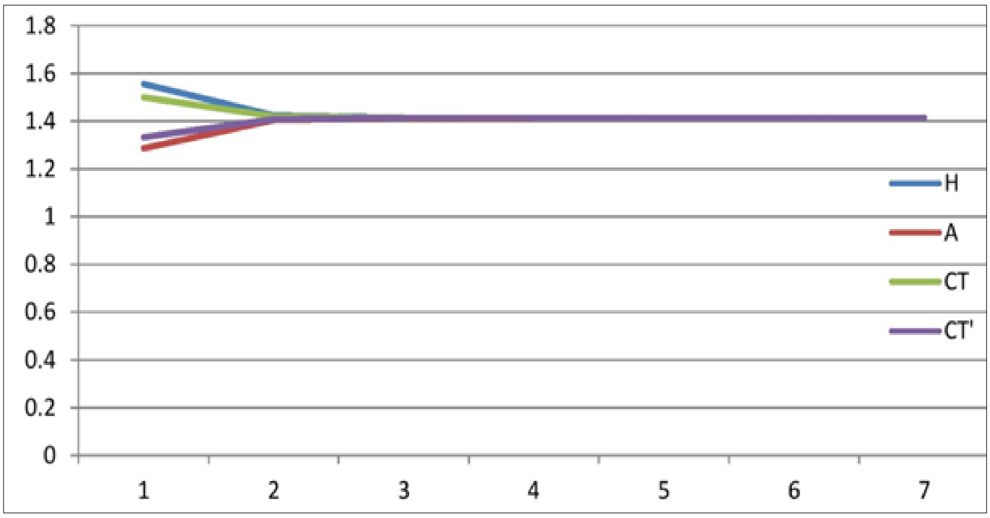


Fig 2: Comparison of Gaussian and centroidal compound means

Table 1: The values of Gaussian compound mean and new compound mean

	Gaussian compound	mean		New Gaussian compound	Mean
	H x A			CT x CT	
N	A_n	B_n	N	A_n	B_n
0	1	2	0	1	2
1	1.5	1.333333	1	1.555555556	1.285714
2	1.420635	1.407821	2	1.424906151	1.403601
3	1.414254	1.414173	3	1.414280427	1.414147
4	1.414214	1.414214	4	1.414213565	1.414214
5	1.414214	1.414214	5	1.414213562	1.414214
6	1.414214	1.414214	6	1.414213562	1.414214
7	1.414214	1.414214	7	1.414213562	1.414214

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