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Identically distributed stochastic integrals, stable processes and semi: Stable processes

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Abstract
This paper establishes conditions under which the identical distribution, to within a shift, of two stochastic integrals defined with respect to the process implies that the process is semistable.

Keywords: Stable processes and semistable processes

Introduction
The main part of the discussion is mostly based on Reidel which concerns itself with stable processes in the same context after identifying some important consequences of the equidistribution, assumption. Here themain result is Theorem.
Also examine two particular cases each in the light of Theorem 1 and independently through more elementary arguments. The first is a stronger version of a result due to Prakasa Rao and Ramachandran [1] and incidentally show that the technical conditions (6) imposed in Theorem 1 while sufficient are not necessary. The second is a strong version of a result due to Lukacs on stable processes given by [2].

Important results and lemma
Let $\{(t),\ t \geq 0\}$ with $X(0) = 0$ be a stochastic process, homogeneous and with independent increments, and continuous in probability. For $j = 1,2$ let $aj\ (\cdot)$.
be a real - valued continuous function and $vj\ (\cdot)$ a non - negative, non - decreasing and right - continuous function defined on the compact interval $[A_j, B_j\ ]$ and let

$$Y_j = \int_{Aj}^{Bj} a_j(t)d\{X(v_j(t))\}\ , j = 1,2 \qquad (1)$$

be stochastic integrals defined in the sense of convergence in probability. Yj has the same probability distribution as

$$Y_j^* = \int_{Cj}^{Dj} t\ d\{X(V_j(t))\}\ , j = 1,2 \qquad (2)$$

for appropriate choices of Cj, Dj and functions Vj. If ϕ is the distinguished logarithm of the characteristic function of $X(1)$, then the log ch.f of Yj or $Yj*$ is given by

$$log\ E\{exp(iuY_j)\} = \int_{Aj}^{Bj} \phi\{u \cdot a_j(t)\}dv_j(t) = \int_{Cj}^{Dj} \phi\{tu\}dV_j(t)\ , j = 1,2$$

Let

$$S(z) = \int_{A_1}^{B_1} |a_1(t)|^z \, dv_1(t) - \int_{A_2}^{B_2} |a_2(t)|^z \, dv_2(t)$$

$$\hat{S}(z) = \int_{A_1}^{B_1} a_1(t)|a_1(t)|^{z-1} \, dv_1(t) - \int_{A_2}^{B_2} a_2(t)|a_2(t)|^{z-1} \, dv_2(t) \quad (3)$$

Let us note that $\hat{s}$ and S are analytic in the open right half - plane $H+$ and continuous and bounded in the closed half - plane $\overline{H}+$.

Let us assume that the Yj are given, for a suitable $Q > 0$ and $Vj\,(\cdot)$ by

$$Y_j = \int_{-Q}^{Q} t \, d\{X(V_j(t))\} \, , j = 1,2 \quad (4)$$

so that (3) takes the form, with $W = V1 - V2$.

$$S(z) = \int_{-Q}^{Q} |t|^z \, dW(t) \, , \hat{S}(z) = \int_{A_1}^{B_1} (sgn\, t)|t|^z \, dW(t) \quad (5)$$

Then make a two- fold classification of the possible cases and a definition based on it

Case 1

S^ ≡ S or $\hat{s} + S \equiv 0$; equivalently, in (5), $dW \equiv 0$ on [- Q, 0) or on (0, Q]

Case 2

$S \pm \hat{s}$ G 0 or dW G 0on [- Q, 0) or on (0, Q]

Let us define

$$G(z) = \begin{cases} S(z) \; in \; case \; 1 \\ S(z) \cdot \hat{s}(z) \; in \; case \; 2 \end{cases}$$

For instance S and $\hat{s}$ are analytic at the origin as well

a) The origin is not a limit - point of the zeros of G in $\overline{H}+$ (6)
b) Lim $x.\, log\, |G(x)| = 0$ $x \rightarrow 0+$

Let us note next that the condition

$$Y_1 \equiv Y_2 + q \quad (7)$$

is equivalent to the following set of four conditions where

(b, σ^2, M, N) is the Levy representation for ϕ, the log ch.f of $X(1)$

$$\int_{-Q}^{Q} [tb + t(1 - t^2) \int_{(0,\infty)} \frac{x^3}{(1 + x^2)(1 + t^2x^2)} d\{M(-x) + N(x)\}] \, dW(t) = -q \quad (8)$$

$$\sigma^2 \cdot \int_{-Q}^{Q} t^2 \, dW(t) \equiv \sigma^2 \cdot S(2) \tag{9}$$

$$L_1(x) := \int_{[-Q,0)} -N(x/t)dW(t) + \int_{(0,Q]} M(x/t)dW(t) = 0 \; for \; x < 0 \tag{10}$$

$$L_2(x) := \int_{[-Q,0)} -M(x/t)dW(t) + \int_{(0,Q]} N(x/t)dW(t) = 0 \; for \; x > 0 \tag{11}$$

Let $r > 0$ be an arbitrary commonpoint of continuity for N and (-).

Suppressing in our notation their dependence on r, let us define

$$H_1(z) = r^z \int_{(r,\infty)} x^{-z} \, dN(x) \quad , H_2(z) = r^z \int_{(-\infty,-r)} |x|^{-z} \, dM(x), \tag{12}$$

$$H = H_1 + H_2, for \; Re \; z \geq 0$$

Thus H's are analytic in $H+$ and continuous and bounded in $\bar{H}+$. If we define

$$\hat{H}_1(z) = r^z \int_{(0,r)} x^z \, dN(x) \quad , \hat{H}_2(z) = r^{-z} \int_{(-r,0)} |x|^{-z} \, dM(x), \tag{13}$$

$H^\wedge = H^\wedge 1 + H^\wedge 2, \; for \; Re \; z \geq 0$

Lemma 1
Let (6), (10) and (11) hold. Then H_1 and H_2 are meromorphic in the whole plane and admit a representation

$Hj \,(z) = Kj \,(z)/(-z)$ for $Re \; z \leq 0, j = 1,2$

for suitable Kj analytic in H- and continuous and bounded in $\bar{H}$-.

For $Re \; z \leq -2$, $Hj \,(z) = -H^\wedge j \,(z)$ for $j = 1,2$ and $Hj \,(z) = -H^\wedge j \,(-z)$.

Further, the of the Hj are located in the open strip $-2 < Re \; z < 0$ and if

$\alpha_{1,j} = sup\{Re \; z: z \; is \; a \; pole \; for \; H_j\}$

$\alpha_{2,} = sup\{Re \; z: z \; is \; a \; pole \; for \; H_j\}$

then the Hj are bounded in $Re \; z \geq \alpha 1,j + \in$ as well as in $Re \; z \geq \alpha 1,j - \in$ for every.
$\in > 0$. Furthermore, the $\alpha n,j$; $n = 1,2,$; $j = 1,2$ are necessarily poles for Hj and the coefficients en,j of the highest negative power in Laurent expansion for Hj around the point $\alpha n,j$ satisfy:

$c1, > 0; (-1)^l j + 1c2,j > 0$

where lj is the multiplicity of the pole at $\alpha 2,j$.

Lemma 2
Let (6) with $G = S$, (10) and (11) hold. Then H is meromorphic in the whole plane and admits a representation

$(z) = K(z)/S(-z)$ for $Re \; z \leq 0$

for suitable K analytic in H- and continuous and bounded in $\bar{H}$-.
All the poles of H lie in the open strip $-2 < Re \; z < 0$ and H is bounded in $Re \; z \geq \alpha_1 + \in$ and in $Re \; z \leq \alpha_2 - \in$ for every $\in > 0$, where

$\alpha 1 = sup\{Re \; z: z \; is \; a \; pole \; for \; H\}$

$\alpha_2 = sup\{Re\ z: z\ is\ a\ pole\ for\ H\}$

α_1, α_2 are themselves necessarily poles for H and the coefficients cn, n = 1,2 of the highest negative power in Laurent expansion for H around the point αn satisfy:

$c1 > 0$; $(-1)^{+1}c2 > 0$

where l is the multiplicity of the pole at α_2.

Theorem 1

Let Y_1 and Y_2 defined by (1) or (2) satisfy the equi - distribution assumption (7) and let (6) be assumed. Then a set of sufficient conditions for

$\{(t)\}$ to be a semi - stable process is given by the following: (i).S has a unique real zero α in the interval (0,2];

i) if (0 <)α < 2, the multiplicity of the zero of S at α is at most two; and
ii) if (0 <)α < 2, then the zeros of $G(z)$ on $Re\ z = \alpha$ form a subset of a set of the form $\{\alpha + in\ \rho: n\ integer\}$ for some $\rho > 0$; α is then the exponent of the semi - stable process.

Conversely, if $q = 0$ or S^(1) $\neq 0$ then conditions (i) - (iii) are also necessary if (9.7) is to be satisfied only by semi - stable processes.

Proof Sufficient Part

Let (6), (7) and conditions (i) - (iii) be assumed to hold. Then let us show that (1) follows a semi - stable law, with the assumed unique zero of S in (0,2] as the exponent. Denote this by α. If $\alpha = 2$, Then in view of Lemma (2),

H is an entire function, and it follows then from the theory of Laplace Stieltjes transforms that the integral representation.

H(z)=$\int_r^\infty \left(\frac{x}{r}\right) - zd\{N(x) - M(-x)\}$ holds for all z now.

Letting $z \to -\infty$ through real values in the above, and noting on the other hand that H is bounded in $Re\ z \leq -2$.

Let us see that $(x) = M(-x) = 0$ for $x \geq r$ and $r > 0$ being an arbitrary common point of continuity of N and M(-.), $N \equiv M \equiv 0$ or $X(1)$ has a normal distribution in this case.

Let now $0 < \alpha < 2$.Then (2) $\neq 0$ by (i) and then (9.9) implies that $\sigma = 0$. For further analysis I this case let us need to consider H_1 and H_2 defined in (12) and the subject of Lemma 1. Let us conclude as argued above for H that if

H_1 G 0, then H_1 has a real pole, and then that this pole must be at -: for this conclusion a little elaboration of the argument in Reidel appears desirable, namely, by Lemma 2, must have $\alpha_1 = \alpha_2(= -\alpha)$ now. Suppose if possible that

$\alpha 21 < \alpha 22$. Then, since $H = H1 + H2$, conclude from Lemma 1 and 2 that

$\alpha 21 = \alpha 2$, so that $\alpha 12 \geq \alpha 22 > \alpha 21 = \alpha 2 = \alpha 1$; then H has a pole at

$max\ (\alpha 11, \alpha 12) > \alpha 1$ again from Lemmas 1 and 2, contradicting Lemma 2. Hence must have $\alpha 21 = \alpha 22$ (= $-\alpha$) and similarly $\alpha 11 = \alpha 12$ (= $-\alpha$).Hence

$\alpha 11 = \alpha 21 = -\alpha$ so that $H1$ can only have poles on the vertical line $Re\ z = -\alpha$.

It then follows from assumption (ii) that $l = max(l1, l2) = 1$ here, in the notation of Lemmas 1 and 2 as in Riedel, so that the multiplicity of the pole for

H_1 at - α is exactly one, and similarly for H_2. Thus if H_1 G 0, then whether α is a simple or a double zero for S, H_1 has a simple pole at - α. Any pole of H_1 on the line $Re\ z = -\alpha$ is then necessarily simple as well, since the integral representation for H_1, valid for $Re\ z > -\alpha$, implies that for all $x >$, real y and integers $n \geq 2$,

$x^n\ |H1(-\alpha + x + iy)| \leq x^n\ H1(-\alpha + x) \to 0$ as $x \to 0 +$

in view of $-\alpha$ being only a simple pole for H_1.

For a suitable $c = (\delta) > 0$,

$$H_1(z) = H_1(0) + \sum_{n=-\infty}^{\infty} a_n \left(\frac{1}{z - z_n} + \frac{1}{z_n}\right) \quad for\ z \neq and\ z_n$$

Also note that, an being the residue of $H1$ at zn

$$a_n = \lim_{x \to 0+} xH_1(x + in\,\rho - \alpha)$$

and therefore $|a_n| \leq a_0$ for all n. Consider now the case $r = 1$ for simplicity. Then invoking the inversion formula for the Laplace transform as conclude that for all $u > 1$

$N(u) = -\xi(u)u^{-\alpha}$

where $0 < \xi(u) = \xi(ue^{2\pi/\rho})$ for all $u > 1$. Replacing $r = 1$ by an arbitrary
$r > 0$, let us see that the above representation holds in fact for all $u > 0$.
Similarly, if $H2$ 0 equivalently if M 0, then $(u) = \eta(u)|u|^{-\alpha}$ for
$u < 0$ with $0 < (u) = \eta(ue^{2\pi/\rho})$ for all $u < 0$.Already noted that $\sigma = 0$.

Thus X_1 has a semi - stable distribution to within a shift, and the sufficiency part of the theorem stands proved.

Necessity Part
Let us proceed to show that if $q = 0$ or $S\^(1) \neq 0$, and if (6) and (7) hold then $\{X(t)\}$ need not be a semistable process if any of the conditions (i) - (iii) is violated. The facts (a) - (d) established below will imply the necessity of these conditions.
For some non - trivial $\{X(t)\}$ to satisfy (7) in the presence of (6),

S must have a zero in the interval (0,2]. If S were to have no zeros at all in (0,2], then (6) and (7) being satisfied, the meromorphic function H has by Lemma 1 no poles anywhere in the complex plane, i.e., it is an entire function. By the theory of L - S transforms, the integral representation for H must hold therefore for all complex z. Let us conclude, letting $z \to -\infty$ through real values and noting that H
Is bounded in $Re\ z \leq -2$ that, in the Levy representation for the log ch.f of

$X(1)$, $M \equiv N \equiv 0$. Hence (1) has a normal distribution but then (9) implies that

(2) = 0, contradicting the assumption that S has no zero in (0,2]. Hence fact (a).

Let us use the assumption that $q = 0$ or $S\^(1) \neq 0$. Let us define

$$b^* = \begin{cases} 0 \ if \ q = 0 \\ -q/\hat{S}(1) \ if \ \hat{S}(1) \neq 0 \end{cases}$$

so that $q + b^*. S\^(1) = 0$ in either case.

If $q = 0$ or $S\^(1) \neq 0$, there cannot be more than one zero for S in the interval (0,2], if (7) is to be satisfied only by a semi - stable process. If possible, let $0 < \alpha < \beta \leq 2$ be two zeros for S in (0, 2]. If $\beta = 2$, consider(b, σ^2, M, N) with $b = b^*$, $\sigma > 0$, $M(-x) = -N(x) = cx^{-\alpha}$ for $x > 0$ for some $c > 0$. Then (8) - (11) hold. If $\beta < 2$ then consider (b, σ^2, M, N) with $b = b^*$, $\sigma = 0$, $M(-x) = -N(x) = c1x^{-\alpha} + c2x^{-\beta}$ for arbitrary $c1, c2 > 0$. Then again (8) - (9.11) hold. Hence fact (b), both the above constructed processes being non - semi - stable.

Facts (a) and (b) establish the necessity of condition (i).
Let α be the unique zero of S in (0,2]. If $\alpha = 2$, as let us have seen, (1) is normally distributed, and the process is trivially semi - stable. Therefore consider the cases (0 <) < 2 in what follows.
Condition (ii) is necessary. Suppose $(\alpha) = S'(\alpha) = S''(\alpha) = 0$. Take

$b = b^*$, $\sigma = 0$, $(-x) = -N(x) = q1 + q2log\ x + q3(log\ x)^2.\ x^{-\alpha}$, $x > 0$ where the qj real are chosen so as to satisfy conditions on M and N for the Levy representation for example $q1 = \alpha^{-3}$, $q2 = \alpha^{-2}$, $q3 = (2\alpha)^{-1}$. Then (8) and (9) hold. To verify (11), coinciding with (10), let us compute for $x > 0$,

$-L2(x) = (q1 + q2\ log\ x + q3\ log^2 x)x^{-\alpha}\ S(\alpha)$

$-(q2 + 2q3\ log\ x)x^{-\alpha}\ S'(\alpha) + q3x^{-\alpha}\ S''(\alpha) \equiv 0$

Hence fact (c)
Finally let us show that if the zeros of G on $Re\ z = \alpha$ do not lie on a lattice with α as a lattice point, then again (7) could be satisfied by a non semi stable process

Let $E_1 = \{\alpha + i\beta n: n = 0,1, \quad \}$ be an enumeration of the zeros of S on

$Re\ z = \alpha$ and let $E2 = \{\mathscr{L}n = \alpha + i\gamma: n = 1, 2, \ldots \ldots\}$ be an enumeration of the zeros of $S^\wedge$ on $Re\ z = \alpha$ not in $E1$. Let us take $\beta 0 = 0 (\alpha \in E1)$ and $\alpha \notin E2$. If $E2$ is empty, let us take $L(b, \sigma^2, M, N)$ according to $b = b^*$, $\sigma > 0$

$(-u) = -N(u) = u^{-\alpha} \{1 + \sum c_n (u^{i\beta}n + u^{-i\beta}n)\}$

where cn and dn below are such that

$\sum|c_n|(1 + |\beta_n|) < 1/4, \sum|d_n|(1 + |\gamma_n|) < 1/4$

so that the requirements that $N \leq 0$ and that N is non - decreasing are satisfied.
If E_2 is non - empty let us take (b, σ^2, M, N) according to $\sigma = 0$ and for

$u > 0$,

$(-u) = u^{-\alpha} \{1 + \sum c_n (u^{i\beta}n + u^{-i\beta}n) + \sum d_n (u^{i\gamma}n + u^{-i\gamma}n)\}$

$-(u) = u^{-\alpha} \{1 - \sum c_n (u^{i\beta}n + u^{-i\beta}n) - \sum d_n (u^{i\gamma}n + u^{-i\gamma}n)\}$

Using the fact that $w \in C$ and real $t \neq 0, \pm 1$ with $\theta = w/2$,

$$\int_0^\infty \frac{x^{2-w}dx}{(1+x^2)(1+t^2x^2)} = \begin{cases} -\pi(1 - |t|w - 1) \sec\theta/\{2(l - t2)\} \text{ if } w \neq 1, \\ -\log\frac{|t|}{1-t^2} \text{ if } w = 1 \end{cases}$$

Let us see that (8) will be satisfied if

$S^\wedge(1)\ \{b + 2\pi \sum d_n Re[\mathscr{L}_n\ sec(\pi\mathscr{L}_n/2)]\} = -q$

$q = 0$ or $\hat{s}(1) \neq 0$ unless all the conditions (i) - (iii) are satisfied, a non - semi - stable process could satisfy (7) holds for some real q only if $\{X(t)\}$ is a semi - stable processed.

Conclusion

The study demonstrates the successful application of the proposed algorithm MHFDA for finding the shortest path in a network model with hesitant fuzzy (HF) costs. By maximizing the HF costs along the selected path, the algorithm effectively identifies the optimal route from node 1 to node 6. The results confirm the utility of hesitant fuzzy logic in handling uncertainty within network models, offering a robust approach for decision-making processes where traditional methods might fall short.

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