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# Differential calculus in modelling predator-prey interactions within ecosystems after the introduction of invasive species

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**Abstract**
The spread of Cheetahs in the Kuno National Park is modelled by a simple diffusion model, which is a partial differential equation, after assuming a system of only species of predator, cheetahs, and one species of prey, deer. It is found the population of cheetahs expands radially outwards from the point they have been released, whilst the population of deer reduces proportional to the increase in cheetah population.

**Keywords:** Lotka–Volterra equations, multivariable calculus, population and spatial dynamics, diffusion model, telegraph process

## Introduction
A major recent event in my country was the reintroduction of cheetahs in India, where a small population of Southeast African cheetahs was brought to the Kuno National Park in Madhya Pradesh, from Namibia (Dhanalakshmi). Reading further on the topic, I found mixed reactions within the scientific community; some zoologists and wildlife biologists praise the reintroduction as supporting conservation efforts and providing a “protected space” for the endangered species, whilst others criticized the efforts as being unsustainable and disruptive to the local ecosystem (Khanwalkar). Such polarization of opinions vastly stem from the high uncertainty in knowing how the newly introduced population would interact with native species and grow within the entire ecosystem.
Therefore, holding a natural affinity for the subject, I decided to explore the application of mathematics in ecological systems to model the spatial interactions of the cheetah population with native species’ populations, which includes how cheetahs prey on and compete with other animals, and their population dynamics, which is the modelling of the rate of change of populations across space and time. Additionally, its strengths and limitations in adapting and applying will be a major focus of this essay.

## Describing the spatial interactions of cheetahs
The spatial interactions of cheetahs are mainly concerned with the movement of cheetahs around an ecosystem and how such movement is affected by numerous factors such as migration, weather, other animals, etc. The following section aims to find the mathematical model for cheetah movement which would most accurately model real world cheetah movements in an ecosystem like that of the Kuno National Park.
To begin with, the movement of cheetahs can be represented as the rate of change of population density, A, with respect to time at varying spatial coordinates. Mathematically, such a model would involve a multivariable function relating *A* with time, and each of the spatial coordinates. To find the movement therefore, a partial derivative of the function must be found with respect to only time, keeping other variables, i.e. the spatial coordinates, constant.
Considering a 2–dimensional space – ignoring the third dimension, height, as it can be reasonably assumed cheetahs move only on a 2–dimensional plane as they are flightless – it is represented as,

$$\frac{\partial A(x,y,t)}{\partial t}$$

Where $A(x,y,t)$ is a function for the population density at spatial coordinates $x$, $y$, and time $t$.
Secondly, cheetahs can initially be assumed to move by a simple diffusion process, specifically Brownian motion, where their motion is completely unpredictable and in random directions at infinite velocities (Holmes 3). Though such conditions are highly unrealistic and no animal could display such movement, it serves as a good starting point for creating more complicated, and realistic, models.
Fick's laws of diffusion, proposed by Adolf Fick, describe motion via diffusion (Philibert 1). His first law, models the evolution of the concentration of a substance, analogous to population density, over time, at particular spatial coordinates. In a 2–dimensional space it gives:

$$\boldsymbol{J} \propto -\nabla(\varphi)$$

$$\therefore \boldsymbol{J} = -D\nabla(\varphi) \quad (1)$$

where $\boldsymbol{J}$ is the diffusion flux vector which measures the rate of flow of a substance across a unit area, $\varphi$ is the concentration of the substance, D is the diffusion coefficient giving the rate of diffusion that is unique to each system and has units $m^2s^{-1}$, and $\nabla$ is the gradient operator giving the gradient in the direction where it is the greatest.
In a closed system, the law of conversation of mass tells us the quantity of the substance in the system remains constant. Mathematically, it is expressed by the continuity expression, which gives the change in quantity of the substance:

$$\frac{\partial\varphi}{\partial t} + \nabla\cdot\mathbf{J} = 0 \iff \frac{\partial\varphi}{\partial t} = -\nabla\cdot\mathbf{J} \quad (2)$$

Where $\nabla\cdot$ is the divergence operator giving the outward flux, or outwards rate of flow, of a substance from an infinitesimal volume around a certain coordinate? Thus, the continuity expression states that the rate of change of concentration of a substance at any point is equal to the negative rate of flow of that substance out from that point per unit area.

Substituting (1) into (2) yields: $\frac{\partial\varphi}{\partial t} + \nabla\cdot -D\nabla(\varphi) = 0$

$$\therefore \frac{\partial\varphi}{\partial t} - D\nabla\cdot\nabla(\varphi) = 0 \quad (3)$$

The operator $\nabla\cdot\nabla$ gives the "divergence of the gradient" and results into the Laplacian, $\Delta$. This separates the rate of change of concentration of the substance with respect to distance in the $x$ and $y$ directions.

$$\nabla\cdot\nabla\varphi = \Delta\varphi \quad (4)$$

The Laplacian, $\Delta$, of a function $f$ in $n$–dimensional space is defined:

$$\Delta f = \sum_{a=1}^{n}\frac{\partial^2 f}{\partial x_a^2} \quad (5)$$

Where $x_a$ are the cartesian coordinates in the $a^{\text{th}}$–dimension.

Substituting (4) into (5) yields: $\frac{\partial\varphi}{\partial t} - D\Delta\varphi = 0$

From (5), taking a 2–dimensional space $\therefore \frac{\partial\varphi}{\partial t} - D(\sum_{a=1}^{2}\frac{\partial^2\varphi}{\partial x_a^2}) = 0$

$$\therefore \frac{\partial\varphi}{\partial t} - D\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) = 0$$

Thus yielding Fick's 2nd law $\therefore \frac{\partial\varphi}{\partial t} = D\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right)$ (6)

Thus the equation gives how the concentration of a substance changes at varying spatial coordinates with respect to time. This partial differential equation can thus be adapted to predict the movement of cheetahs: $\varphi$, the concentration of a substance, is analogous to the population density $A$ in a system of cheetahs instead of the concentration of a substance $\varphi$, giving,

From (6), $\frac{\partial A}{\partial t} = D\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right)$ (7)

Therefore, this partial differential equation (PDE) is a model for the rate of change of population density at each spatial coordinate. $D$, the diffusion coefficient, measures the rate of diffusion which is unique to each animal/species in each ecosystem. The solution for this PDE is the function relating $A$ to the spatial coordinates $x$, $y$, and time $t$, and was graphed, using *Mathematica*, for an ecosystem where an arbitrary number of organisms were released with population density $A = 1$ at spatial coordinates $(0, 0)$ at $t = 0$ (Figure 1.). Population density is only shown on the $x$ axis as it would be impossible to depict 4 variables and the population density evolves the same in both axes thus it is satisfactory as we are only interested in understanding how cheetahs would disperse from the center.

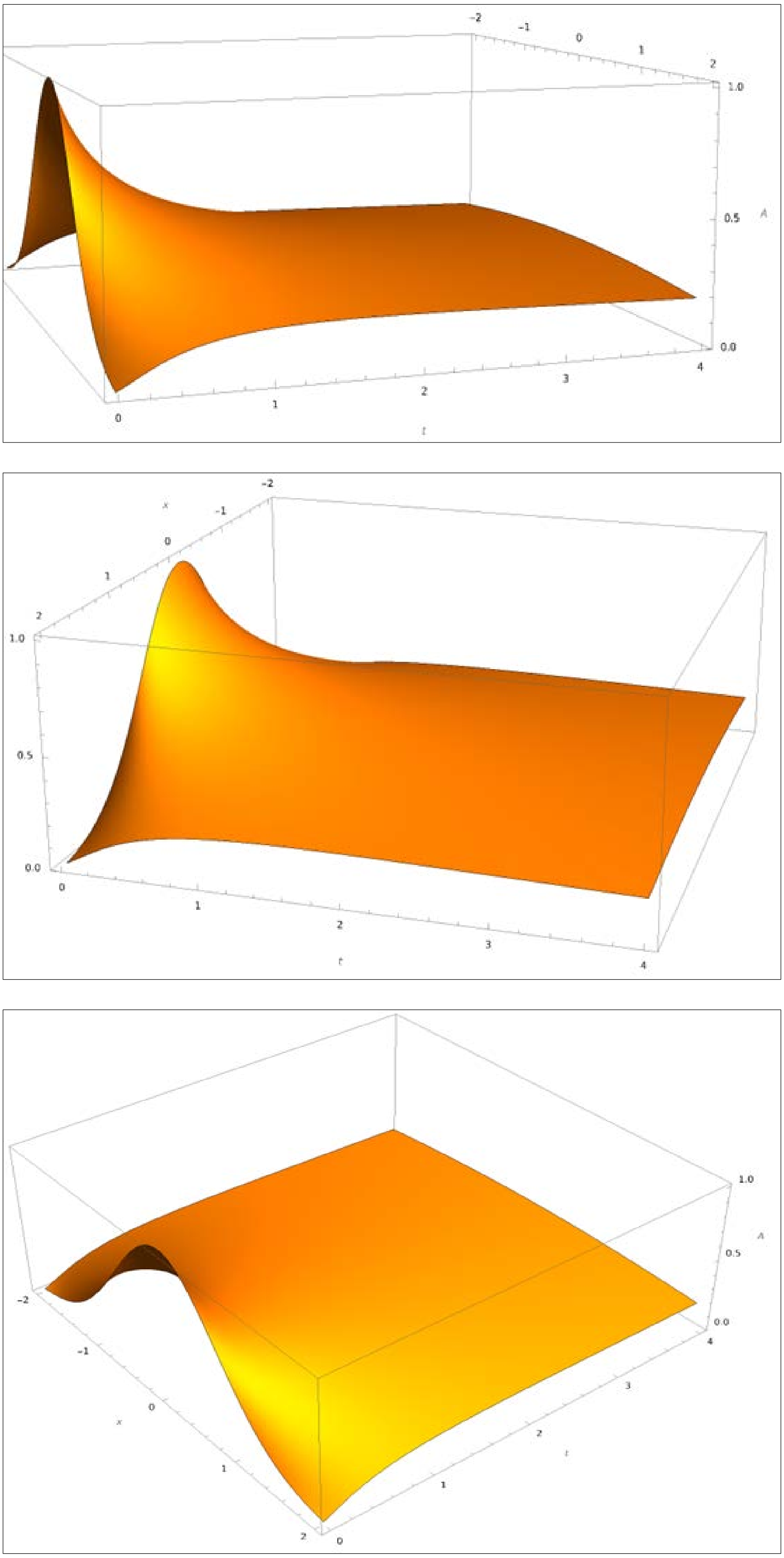


**Fig 1:** Graph showing the relationship between $A$, $x$, and $t$.

Thus, we can make an immediate observation from the graph: at $t = 0$, the population density is exactly 1 at $x = 0$, and very concentrated near $x = 0$ at low values of $t$ where the cheetahs are assumed to have been released.

Furthermore, it is also seen that as $t \to \infty$, $A$ tends to become evenly distributed across all values of $x$, showing that over time, the distribution of cheetahs tends to become homogenous. In fact, at each point in time, the position of the cheetahs are normally distributed and thus $A$ is given by a bell shaped curve which flattens over time indicating the cheetahs getting more equally distributed.

These results serve as a sanity check, as it is expected that the cheetahs moving randomly would eventually tend towards an even distribution thus indicating the model logically makes sense when applied.

However, there are weaknesses to this model. Firstly, unlike one would expect, $A$ at all other coordinates is not 0 and rather is an extremely low, non–zero positive value. As a matter of fact, at all $t$, the population density $A$ never reaches 0, implying some cheetahs can move extremely large distances in very short periods of time. This is due to the underlying assumption that cheetahs move at infinite velocities. However, given the drastic decrease in $A$ as $x$ gets really high for small values of $t$, the model only predicts an extremely small, negligible probability of cheetahs to be able to move impossibly far away. Thus, given that we are only concerned with the average behavior of cheetahs, not improbable one off events, such a shortcoming does not affect the applicability of the model.

Another shortcoming of this model is that it assumes cheetahs have completely random motion, on both small and large time scales, which is not true. On short time scales, cheetahs have momentum and inertia and thus cannot arbitrarily change their directions instantaneously, and instead, would resist such changes and continue to move in the same direction for long periods of time. Although, this would likely not make a difference on our results as we are only interested in the average movement of cheetahs over longer periods of time.

Although, even on long time scales, most animals, including cheetahs, tend to move in certain specific directions depending on the climate, habitat, human intervention, etc. Therefore, the outcome that all cheetahs become equally spread out over long periods of time is unrealistic, as they tend to be more concentrated in some areas than others. Usually though, these effects are minor, and the overall movement of animals can still be assumed to be by simple diffusion

Moving onwards, it is also important to derive a function giving the mean positions of cheetahs at any time $t$ for modelling the spread of cheetahs, which in this analysis may be more useful than finding the likely position of every cheetah. Thus, in this next section, a function is found relating the mean position of the cheetahs with respect to $t$ along the $x$ direction, position being the displacement from the origin along the $x$ axis. Once again, because the spread of cheetahs is the same in both directions, finding the displacement in one direction is sufficient.

In this case, cheetahs move equally in the positive and negative directions and thus simply summing the displacements of all cheetahs would yield an average displacement of 0. Thus, the root–mean–squared (RMS) displacement along the $x$ axis at time $t$, $\alpha_x(t)$, would be a better measure of displacement over time (RMS displacement being the square root of the sum of the square of all displacements).

First, let there be $\rho$ cheetahs released at the origin, $x = 0$ m, with $X_c(t)$ being the position of the $c^{\text{th}}$ cheetah after $t$ seconds. Thus, the sum of the square of all displacements at $t$ is,

$$X_1(t)^2 + X_2(t)^2 + X_3(t)^2 \ldots x_\rho(t)^2 = \sum_{c=1}^{\rho} X_c(t)^2$$

$$\therefore \alpha_x(t) = \sqrt{\frac{1}{\rho}\sum_{c=1}^{\rho} X_c(t)^2} \quad (8)$$

The displacement of a cheetah $\beta$ m in 1 second is given by,

$$\pm\beta = X_c(t-1) - X_c(t) \Rightarrow X_c(t) = X_c(t-1) \pm \beta$$

The $\pm$ sign is included because roughly half the cheetahs move in the positive direction while the other half moves in the negative direction. In the next step, the displacement of the cheetah at $t$ is squared,

$$X_c(t)^2 = (X_c(t-1) \pm \beta)^2 = X_c(t-1)^2 \pm 2\beta X_c(t-1) + \beta^2 \quad (9)$$

Thus, the mean–squared displacement (MSD) is, (from (8) and) (9)

$$\alpha_x(t)^2 = \frac{1}{\rho}\sum_{c=1}^{\rho}(X_c(t-1)^2 \pm 2\beta X_c(t-1) + \beta^2)$$

$$= \frac{1}{\rho}\sum_{c=1}^{\rho} X_c(t-1)^2 \pm \frac{1}{\rho}\sum_{c=1}^{\rho} 2\beta X_c(t-1) + \frac{1}{\rho}\sum_{c=1}^{\rho}\beta^2 \quad (10)$$

From (9), it is evident the first term, $\frac{1}{\rho}\sum_{c=1}^{\rho} X_c(t-1)^2$ is simply the MSD at time $t-1$. Thus,

$$\alpha_x(t-1)^2 = \frac{1}{\rho}\sum_{c=1}^{\rho} X_c(t-1)^2 \quad (11)$$

The next term will average out to zero, as half the cheetahs have positive displacements and half have negative displacements.

$$\pm\frac{1}{\rho}\sum_{c=1}^{\rho} 2\beta X_c(t-1) = 0 \quad (12)$$

Thus, from (10), (11) and (12),

$$\alpha_x(t)^2 = \alpha(t-1)^2 + \frac{\beta^2}{\rho} = \alpha(t-1)^2 + \delta^2 \quad (13)$$

Where $\delta^2 = \frac{\beta^2}{\rho}$, with units $m^2$.

From this result therefore, it is evident that the MSD $\alpha(t)$ increases every second by $\delta^2$. Since all cheetahs have zero displacement initially, this means that $\alpha(t)$ is a linear function with gradient $\delta^2$.

$$\alpha_x(0)^2 = 0$$

$$\alpha_x(1)^2 = \alpha_x(1-1)^2 + \delta^2 = \delta^2$$

$$\alpha_x(2)^2 = \alpha_x(2-1)^2 + \delta^2 = \delta^2 + \delta^2 = 2\delta^2$$

$$\therefore \alpha_x(t)^2 = t\delta^2 \quad (14)$$

Furthermore, since $\delta^2$ gives a measure of the additional area covered by the cheetahs per unit $t$, it is equivalent to a measure of rate of diffusion. Thus, it is related to the diffusion constant $D$ by,

$$D = \frac{\delta^2}{2} \Rightarrow \delta^2 = 2D$$

$$\therefore \alpha_x(t)^2 = 2Dt \quad (15)$$

Thus, the RMS displacement is,

$$\therefore a_x(t) = \sqrt{2Dt} \quad (16)$$

The MSD in 2 dimensions $b(t)^2$ can be found by adding the component MSDs in each dimension,

$$\therefore b^2(t) = a_x(t)^2 + a_y(t)^2,$$

Where $a_y(t)$ is the RMS displacement along the $y$ axis. However,

$$a_x(t)^2 = a_y(t)^2$$

$$\therefore b(t)^2 = 2 \times a(t)^2 = 2 \times 2Dt = 4Dt \quad (17)$$

$$\therefore b(t) = \sqrt{4Dt} = 2\sqrt{Dt} \quad (18)$$

Therefore, the average displacement of the cheetahs from the origin is proportional to the root of $t$ elapsed. The rms displacement is equivalent to the standard deviation $\sigma$ of the position of the cheetahs as both are equal to the square root of the sum of the squares of the deviations of position of every cheetah at time $t$.

$$a_x(t) = \sigma_x$$

$$b(t) = \sigma$$

Where $\sigma_x$ is the standard deviation of the position of cheetahs along the $x$ axis.
This allows us to find the exact solution for the PDE (7) in 1 dimension. Since the population densities $A$ of cheetahs along the $x$ axis is normally distributed at every $t$, it is of the general form of a normal distribution curve $f(x)$ with mean $\mu$ and standard deviation $\sigma$,

$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2}$$

$$\therefore A(x,t) = \frac{1}{a_x(t)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{a_x(t)}\right)^2} = \frac{1}{\sqrt{2Dt}\times\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{2Dt}}\right)^2} = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \quad (19)$$

The average velocity of the cheetahs, $v(t)$ ms$^{-1}$, can also be calculated from (18) by differentiating the RMS displacement with respect to $t$.

$$v(t) = \frac{d}{dx} 2\sqrt{Dt} = 2\frac{d}{dx}(Dt)^{\frac{1}{2}}$$

$= 2 \times \frac{1}{2}(Dt)^{-\frac{1}{2}}$ (Using the chain rule of differentiation)

$$\therefore v(t) = \frac{1}{\sqrt{Dt}} \quad (21)$$

The average velocity of cheetahs therefore varies inversely with the square root of time $t$. Using these results, it is now possible to further refine our model for the movement of cheetahs.
Firstly, because cheetahs spread out with the same velocity, $v(t)$, in all directions, the population of cheetahs disperse radially outwards from the origin as a wave moving with velocity $\frac{1}{\sqrt{Dt}}$. In this case, the peak of this wave gives the mean position of every cheetah. How the velocity of this wave varies with the distance travelled, can be found by solving for the velocity with respect to RMS displacement, $v(b)$.

$$b(t) = 2\sqrt{Dt}$$

$$\therefore \left(\frac{b(t)}{2}\right)^2 = Dt \Rightarrow t = \frac{b(t)^2}{4D}$$

From (19), $v(t) = \frac{1}{\sqrt{Dt}}$

$$\therefore v(b) = \frac{1}{\sqrt{\left(\frac{b(t)}{2}\right)^2}} = \frac{1}{\frac{b(t)}{2}} = \frac{2}{b(t)} \quad (22)$$

Thus, the velocity of the wave varies inversely with displacement from the origin. This can be represented on a graph as concentric circles, giving the mean position of the cheetahs at time $t$, moving out from the origin as $t$ increases. The graph's equations are thus those of a circle,

$$(x-h)^2 + (y-k)^2 = r,$$

Where $(h,k)$ are the coordinates of the center, and $r$ is the radius of the circle. In this case, $(h,k) = (0,0)$, and the radii of these circles, $r$, is equal to $2\sqrt{Dt}$. Thus,

$$x^2 + y^2 = 2\sqrt{Dt}.$$

Assuming $D = 1$, this can be graphed as,

The velocity with which each point on the circle moves outwards perpendicularly is given by $v(t)$ or $v(b)$, where $b$ is the radius of the circle. Therefore, the spread of cheetahs in the ecosystem can now be modelled, alongside a general description of the likely position of every cheetah in the ecosystem.

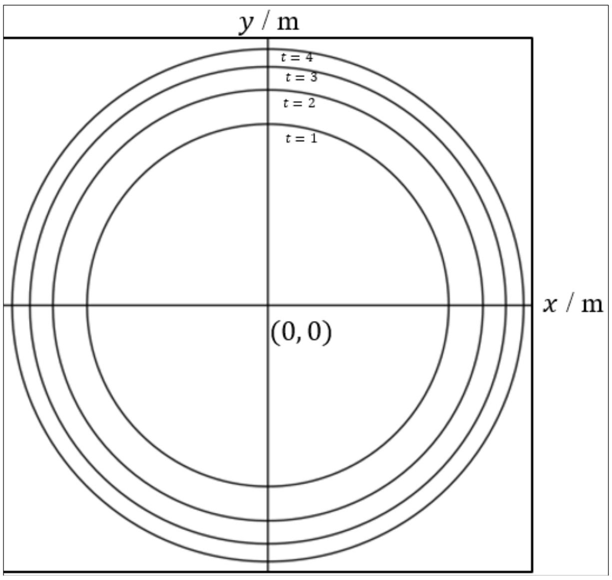


**Fig 2:** Graph showing average RMS displacements along $x$ and $y$ axes of cheetahs at varying $t$

**Describing the population dynamics of cheetahs**

Population dynamics refers to the mathematical modelling of how populations change over time, mostly due to predator and prey interactions. This section is mainly concerned with modelling the rates of change of population densities of cheetahs and their prey, mostly spotted deer, with respect to $t$ at any point $(x, y)$.

To begin with, the rate of changes of population of prey can be modelled. It is assumed the cheetahs only consume one type of prey, which is reasonable as cheetahs majorly consume only spotted deer in the Kuno National Park (Koshy). Here, the instantaneous rate of change of deer population can be represented as the derivative of the deer population density (At a particular spatial coordinate), $u$, with respect to $t$, which is equal to the birth rate, $P_u$, minus the death rate, $D_u$,

$$\therefore \frac{du}{dt} = P_u - D_u \tag{23}$$

The birth rate of deer, which is the number of new deer born, will, under the assumption that the deer have an unlimited food supply and thus can reproduce infinitely, be proportional to the deer population,

$$P_u \propto u$$

$$\therefore P_u = \chi u \tag{24}$$

While the death rate, called the predation rate as it is due to predation, will be proportional to the product of the cheetah population $c$ and the deer population density $u$. This can be seen because a greater number of cheetahs would mean they consume more deer, while a greater deer population mean more are available for cheetahs to eat and thus they eat more.

$$D_u \propto uc$$

$$\therefore D_u = \psi uc \tag{25}$$

Therefore from (23), (24), and (25),

$$\frac{du}{dt} = \chi u - \psi uc \tag{26}$$

Next, the rate of change of population density of cheetahs (at a particular spatial coordinate), which is the derivative of $c$ with respect to $t$, can be modelled by a similar process.

$$\frac{dc}{dt} = P_c - D_c \tag{27}$$

Where $P_c$ is the growth rate of cheetahs and $D_c$ is the death rate of cheetahs. Because cheetahs only grow depending on their available nutrition source, it is proportional to the predation rate,

$$P_c \propto uc$$

$$P_c = \Psi uc \tag{28}$$

Because more cheetahs will imply greater competition for the limited prey (the spotted deer), a greater number of foxes leads to more deaths and so the death rate is proportional to the cheetah population,

$$D_c \propto c$$

$$D_c = \mathrm{X}c \tag{29}$$

Therefore from, (27), (28) and, (29)

$$\frac{dc}{dt} = \Psi uc - \mathrm{X}c \tag{30}$$

Therefore, a pair of differential equations, (26) and (30), have been derived, together known as the Lotka–Volterra predator–prey model (Josef and Sigmund 2). These equations show us that the rate of change of either species' population density is linked with the population density of the other species. These equations together can be used with the earlier spatial modelling to see how the populations of both predators and prey change over time at each spatial coordinate. To do this, it is first necessary to solve these differential equations to attain a deeper understanding of how exactly the populations of both species, predator and prey, are related. This can be done by plotting solutions parametrically on a phase plane, where the $x$ axis represents $u$, and the $y$ axis represents $c$. The phase plane, therefore, shows the trajectory of each species' population (or in other words, the way each population changes) depending on the population of the other. It can therefore be thought of as a vector field where each point is assigned a vector giving the rate of change of both populations. For the following study therefore, the proportionality constants can be eliminated as specific quantities are not being considered, rather only the general trend is being found. This is done by nondimensionalization, which research shows can reduce the Lotka–Volterra equations to,

$$\frac{du}{dt} = u - uc$$

$$\frac{dc}{dt} = \Upsilon uc - \Upsilon c$$

To begin with, vectors can be assigned to both axes. Considering the $x$ axis first, where $c = 0$, the rate of change of population density of $u$ is (from (26)),

$$\frac{du}{dt} = u - u \times 0 = u$$

This therefore eliminates the death rate, indicating that $u$ increases exponentially. Thinking about this biologically, it makes sense because in the presence of 0 predators, the deer can continuously reproduce. Next, considering the $y$ axis, here $u = 0$,

$$\frac{dc}{dt} = \Upsilon 0 \times c - \Upsilon c = \Upsilon c$$

This eliminates the growth rate of cheetahs, indicating $c$ continuously decreases. This again makes sense biologically as without any prey, the cheetahs starve or migrate elsewhere.

However, there can also exist points of stability, called steady states, where neither species' population density has any tendency to change. This is when $\frac{du}{dt} = 0$, and $\frac{dc}{dt} = 0$, and therefore,

$$\frac{du}{dt} = u - uc = 0 \Rightarrow u(1 - c) = 0$$

$\therefore u = 0$, or $c = 1$

If $u = 0$,

$$\frac{dc}{dt} = -\Upsilon c = 0$$

$\therefore c = 0$

If $c = 1$,

$$\frac{dc}{dt} = \Upsilon \mathrm{u} - \Upsilon = 0 \Rightarrow \Upsilon(\mathrm{u} - 1) = 0$$

$\therefore u = 1$ as $\Upsilon \neq 0$

Therefore, the two steady states are at $(0, 0)$ called extinction as both species have 0 populations, and $(1, 1)$ called the center. A stability analysis can be conducted to show that the dynamics of both populations can be described by elliptical phase paths around the center. Depending on the initial conditions, the system of $u$ and $c$ always remains on this path and keep "orbiting" the center on these phase paths. The direction in which the system moves can be found by considering a point on the phase path where $c > 1$, hence, $uc > u$. Because the rate of change of $u$ is given by,

$$\frac{du}{dt} = u - uc$$

It is immediately seen that $\frac{du}{dt}$ will be negative, and $u$ will be decreasing. Therefore, on the phase path, when the system is above the center $(1, 1)$, it must be moving leftwards. The phase plane can now be drawn using *Matlab*,

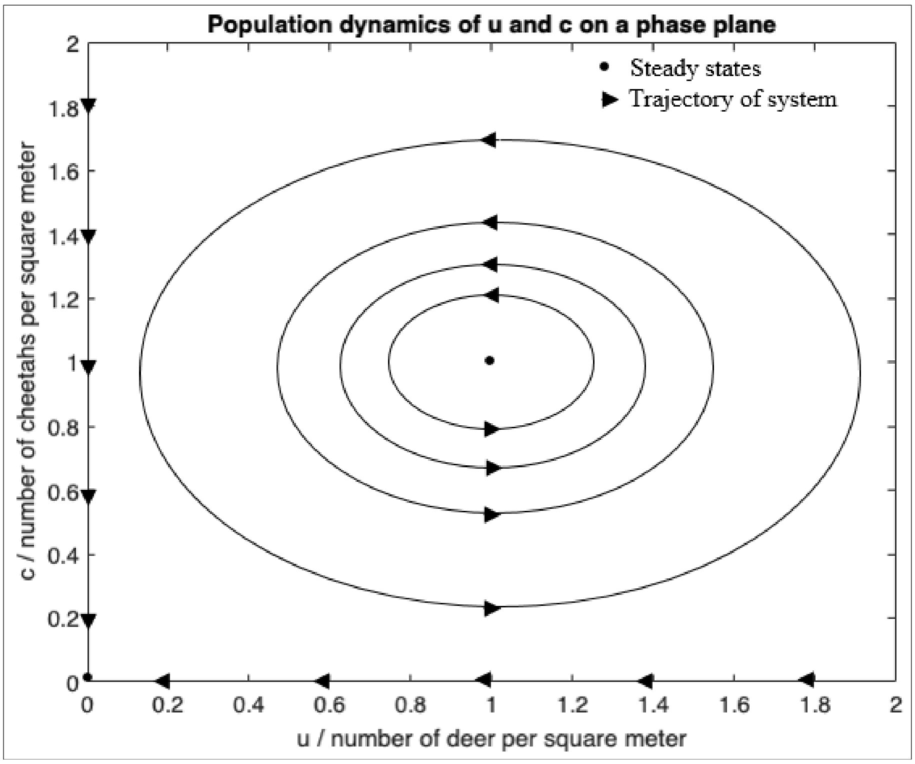


**Fig 3:** Phase plane showing the population dynamics of deer and cheetahs

Considering any initial condition, a starting value for $u$ and $c$, therefore, the evolution of the population density of cheetahs and deer at any spatial coordinate can be determined. However, there are key weaknesses to this model which must be considered.

Firstly, the model assumes that deer are able to find sufficient nutrition and space at all times to keep growing, and thus can reproduce freely in the absence of any predation. However, this is highly unrealistic as in the real world there are limits to the population of prey due to finite

food and space. Although is unlikely to have a major effect on the results of the model as the population of prey rarely ever reaches the point of food or space shortage.
Secondly, it is also assumed that cheetahs only prey on deer which is untrue. However, because studies have shown that spotted deer make up a large majority of a cheetah's diet, this too is mostly a reasonable assumption.

**Predicting the spatial and population dynamics of Kuno National Park after the reintroduction of Cheetahs**
Therefore, finally, both the spatial, and population dynamics can be used together to create a comprehensive description of how the ecosystem would react with the introduction of cheetahs in Kuno National Park.
Firstly, when the 20 cheetahs are released at the origin, (0,0), they spread out by a simple diffusion process, slowly spreading outwards and becoming more evenly spread out throughout the entire ecosystem, as shown by Figure 1. This "spreading" of cheetahs can be measured by determining the how the mean position of every cheetah evolves over time. This is given by (18) and (21), and shows that the cheetahs continuously move outwards, occupying new regions further away from the origin. From the Lotka–Volterra model derived earlier, and the phase plane in Figure 3, it is seen that when the cheetah population density c suddenly increases, the system will be above the steady state, and thus d begins to decrease. Therefore, a travelling wavefront of high cheetah population density expands radially outwards, invading territory previously occupied by prey such as the spotted deer and causing their population to decrease. This can be represented graphically on the x axis (which is identical to the y axis).

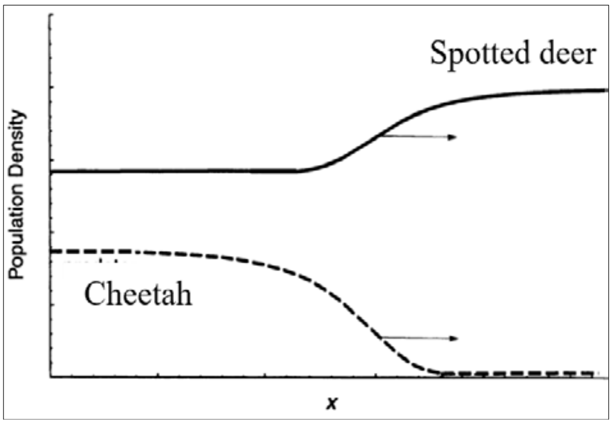


**Fig 4:** Graph showing population dynamics of cheetah and deer (Holmes 5)

Here, the rightward arrows show how the wave front moves further from the origin over time. In the long–term however, the phase plane shows that $c$ eventually begins to decrease as they have fewer and fewer deer to prey on. When this happens, $u$ then begins to increase causing $c$ to increase once again and leading to a continuous cycle of oscillating population densities. This can once again, be shown graphically, using *Desmos*, by considering the initial conditions of high $c$ and decreasing $u$ – just after the cheetah's reach the new region.

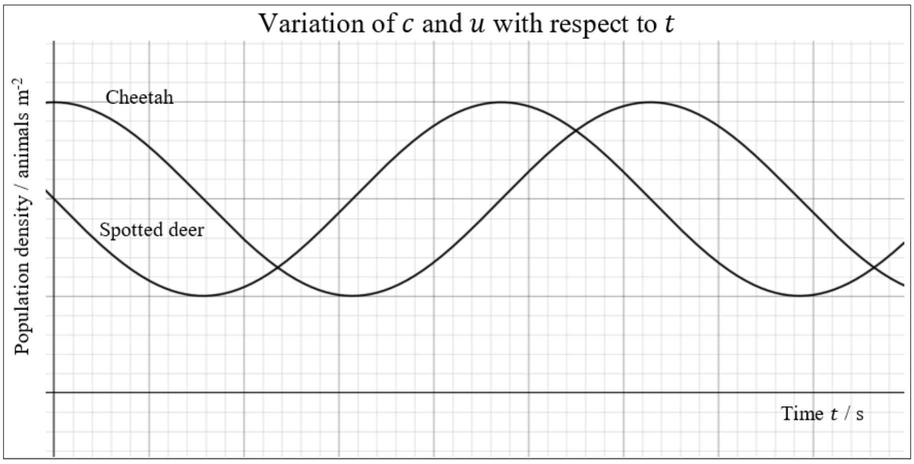


**Fig 5:** Graph showing how the population densities of cheetahs and spotted deer vary with $t$

Therefore, it is seen that the introduction of cheetahs, while introducing instability in the ecosystem, with population densities of both animals continuously changing, would still allow cheetah populations to establish themselves over the long term.

## Conclusion
The aim of the essay was to use differential equations to model both the population, and spatial dynamics of the Kuno National Park's ecosystem following the reintroduction of cheetahs. While models were able to be created to describe how the cheetahs will spread out over the entire ecosystem, and how their population interacts with their primary prey, spotted deer, a major limitation was that only cheetahs and spotted deer were considered in the system. In reality however, there are many other animals and both the cheetah and spotted deer are a part of much more complicated systems, such as a huge interconnected food web. Therefore, limiting the scope of the investigation to only two animals was a major downfall. However, knowing that cheetah's mainly only prey on, and interact directly with cheetahs only, the final conclusion that the cheetahs would still be able to survive, found by differential calculus, still holds true.

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