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Continuity aspects in dynamical systems: Lift maps vs. circle maps

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Abstract
In this study, we thoroughly investigated circle maps and lift maps and defined continuous functions for circle maps and their lifts. Some findings regarding the function and its continuity have been demonstrated. Additionally, some findings based on the continuity of the quotient functions are expanded. Additional findings on the composition of circle maps with lift maps and the proof of the continuity of the connected spaces were also made. Set theory-based fundamental conclusions have been included, which could help in understanding common ideas. Detailed research has also been conducted using lift maps and circle maps that combine lift maps.

Keywords: Circle maps, lift maps, π function, dynamical systems, continuity of functions and composition of functions

1. Introduction
Circles play an important role in whatever study they are involved in. The study of circle dynamics also explains the qualitative characteristics of associated functions, particularly the continuity aspects. In every study they are a part of, circles are significant. When discussing circles in IR^2, the unit circle, or the circle with radius 1, denoted by S^1, is the only one that is considered. This circle in IR^2 represents the perimeter of the closed unit disc. Moreover, S^2 indicates the closed unit sphere border at the origin in IR^3. Similarly, in IR^{n+1}, S^n represents the closed unit sphere boundary at the origin. The main focus of this paper is limited to only IR^2.

The unit circle, or the circle at origin with radius 1, which is represented by S^1, was the primary focus of our investigation. To clarify, we can write f: $S^1 \longrightarrow S^1$ to represent a circle map, which is a self-map on S^1. A circle map's points are moved around the circle according to the map's specifications. The continuity of the circle map is presumed in order for the close points of S^1, the domain of a circle map, to stay close when mapped into S^1, the co-domain of the circle map. The study of dynamics focuses on how a body moves or, more generally, how a system changes over time. In circle dynamics, we study how the points on a circle move, and we do this by testing a function that is defined on the circle.

To get some meaningful analysis related to circle maps, we need to follow some methodology. This includes the assumption of the function π: IR $\longrightarrow IR^2$, whose range is contained in S^1 i.e. π: IR $\longrightarrow S^1$. Then there are lifts of circle maps. The assumption of continuity of F (lift map) appears to be a stringent supposition especially in the presence of the circumstances F is expected to satisfy. Therefore, it ought to be worth studying various properties of lifts of circle maps without assuming continuity of the lift map. Even so, there exist certain functions related to circular dynamics whose continuity has importance. This paper's contents are derived from an analysis of these two varieties of algebraic structure.

Motion occurs through a circle's points in the study of circle dynamics. A function that is defined on a circle is used to attempt to depict this movement of points. Circle maps are a type of such functions. Knowing about the continuity of the relevant functions is part of the study of circle dynamics. A function's continuity at a given location ensures that the function maps the nearby points as closely as possible to each other. The dynamics of circles and circle maps have been studied by many researchers.

This paper has the following layout: Section 2, some basic definitions related to circle maps, π function, lift maps and their continuity are included. Section 3, some further extended results on circle maps and its lifts are illustrated which includes π function and its continuity properties also included. Following this, section 4, continuity of the lift maps theory and some findings related to lift maps are given involving the analysis of the lift map's continuity aspect and its composition with circle maps are presented. Further, section 5, the concept of composition of circle map with lift maps is elaborated and some fundamental results have been given. Section 6, conclusions of the obtained results has been demonstrated. In the end section 7, references have been added.

2. Preliminaries

In this section of paper, we have included the basic definitions of circle map and some other background to define the mathematical structure of circle map. Lift of circle map is a continuous function, F: IR ⟶ IR, satisfying certain conditions which are explained as follows:

1) There is, k ∈ Z this way F(x+1) = F(x) + k for each x ∈ IR and
2) π o F = f o π.

Further, we present various properties of lifts of circle maps without assuming continuity. A circle map is a continuous function f: $S^1 \to S^1$. Let I = [0, 1] and for x ∈ IR, e^{ix} = cosx + i sinx, a periodic function that exhibits periodicity 2π. There is a mapping (covering map) π: IR → C that is described as

$\pi(x) = e^{2\pi ix} = \cos 2\pi x + i \sin 2\pi x$.

Just to mention IR^2, the function π(x) can be written in the form of ordered pair i.e.

π(x) = (cos2πx, sin2πx). Therefore, π is a function from IR to S^1. As π(x + n) = π(x) for every integer n, thus □ is a periodic function with periodicity 1. Suppose for every integer n, π: [n, n + 1) → S^1 is one-one and onto. In particular, π: [0, 1) → S^1 is one- one and onto. If we identify 0 and 1 of [0, 1] then S^1 can be identified with [0, 1].

The concept of a function π plays a significant role in the study of dynamics on circle maps. For the function π, it is sufficient to know the values of π for only the points of [0, 1) as π: [0, 1) ⟶ S^1 can be extended to the whole of IR. We define f_ω: $S^1 \to S^1$ as f_ω(cosx, sinx) = (cos(x+ω), sin(x+ω)), for a fixed ω, 0 < ω < 2π. Here f_ω is a circle map. Correspondingly, if we specify, f_ω^*: $S^1 \to S^1$ specified as f_ω^*(cosx, sinx)

= (cos(x+2πω), sin(x+2πω)), is likewise a circle map. Now, for x ∈ IR, F: IR → IR, define as F(x) = x + ω/2π, then π(F(x)) = π(x + ω/2π) = (cos2π(x+ω/2π), sin2π(x+ω/2π)) = (cos(2πx+ω), sin(2πx+ω)). Now $f_\omega(\pi(x))$ = f_ω(cos2πx, sin2πx) = (cos(2πx+ω), sin(2πx+ω)) gives π o F = f_ω o π this indicate that F is a lift of the circle map $f_□$. Again, we define F^*: IR → IR, $F^*(x)$ = x + ω, for every, x ∈ IR, then $\pi(F^*(x)) = \pi(x + \omega) = (\cos 2\pi(x+\omega), \sin 2\pi(x+\omega))$ and $f^*(\pi(x))$ = (cos2π(x+ω), sin2π(x+ω)). Thus F^* is a lift of the circle map f_ω^*. Since S^1 = {(cos2πx, sin2πx): x ∈ IR}, the function π, maps IR onto the circle $S^1 \subset IR^2$, or C. Also, π: [n, n+1) → S^1 is one-one and onto, for every integer n. In particular, π: [0, 1) → S^1 is one-one and onto.

3. Circle map and its lift

A circle map can be defined as a continuous function f: S^1 → S^1 and its lift can be thought of as a continuous function F: IR ⟶ IR.

Remark 3.1: Let g: X⟶ X, then for every n ∈ IN,

i) $g^{n+1} = g^n$ o g
ii) g^n o g = g o g^n
iii) Consider that H ⊂ IR and g: H ⟶ [0, 1) ⊂ IR. Then, π o (g + n) = π o g for every n ∈ Z.

Proof. (i) We prove this result by induction. For n = 1, we have g^2 = g o g, which is obvious. Now suppose the result is true for m i.e. $g^{m+1} = g^m$ o g. Now we prove this for n = m+1, therefore we have g^{m+2} = g o g^{m+1} = g o (g^m o g) = (g o g^m) o g = g^{m+1} o g.

i) It follows by (i) and the definition of g^{n+1}.
ii) Let the composition maps are, π o (g + n): H ⟶ S^1 and π o g: H ⟶ S^1. Suppose, x ∈ H. (π o (g + n))(x) = π((g + n)(x)) = π(g(x) + n) = π(g(x)) = (π o g)(x).

Remark 3.2: Let F: IR ⟶ IR then,

i) For every, n ∈ IN, π o (F + n) = π o F.
ii) Let f: S^1⟶ S^1 be a circle map. If π o F = f o π, then F = f^* on [0, 1).
iii) Assume a constant k exists such that, F(x + 1) = F(x) + k, for every, x ∈

IR, then F(x + n) = F(x) + nk, for every n ∈ Z. Proof. (i) It follows directly by using Remark.3.1. (iii).

1) It follows directly by the definition of f^*.
2) This can be proved by using induction on n ∈ IN, and taking n = – m, for some m ∈ IN.

Remark 3.3: Let us consider f: S^1 ⟶ S^1 be a circle map. If F: IR ⟶ IR is a lift for f. Thus, for every n ∈ Z, F + n is also a lift for f.

Proof. For every x ∈ IR, there is some value of k in Z such that F(x + 1) = F(x) + k. Consider if, F: IR ⟶ IR then one can easily see that F + n: IR ⟶ IR is continuous.

Suppose if, x ∈ IR we have, (F + n)(x + 1) – (F + n)(x) = (F(x + 1) + n) – (F(x) + n) =

F(x + 1) + n – F(x) – n = F(x + 1) – F(x) = k. By using, Remark 3.2. (i), π o (F + n) = π o F. Now by definition of lift, π o F = f o π. Therefore, π o (F + n) = f o π. As a deduction, F + n is also a lift for f.

Remark 3.4: Let f: S^1 ⟶ S^1 be a circle map and F: IR ⟶ IR. Then,

i) F = f^* on [0, 1).
ii) F([0, 1)) ⊂ [0, 1).
iii) π o F = f o π.
iv) F(x + 1) = F(x) + 1 for each x ∈ IR.

Proof. Think of a mapping, f^*: [0, 1) ⟶ [0, 1). Assuming that, x ∈ IR and x = [x] + r(x). Let's define, F(x) = [x] + f^*(r(x)).

i) It is obvious from there that, F = f^* on [0, 1).
ii) Consequently, F([0, 1)) ⊂ [0, 1).
iii) Considering that, x ∈ IR and r(x) ∈ [0, 1).

We have, (π o F)(x) = (π o f^*)(r(x)) = (f o π)(x). Hence the outcome.

iv) Considering each x ∈ IR, F(x + 1) = [x + 1] + f^*(x + 1 – [x + 1]) = [x] + 1 + f^*(r(x)) = [x] + f^*(r(x)) + 1 = F(x) + 1. Thus the outcome follows.

4. Continuity of lift of circle maps

Lift of a circle map is a continuous function by definition, F: IR ⟶ IR satisfying certain conditions. Usually, we assume that F is continuous. But up to a point we can consider lift of a circle map without assuming continuity of circle map. We obtain below some properties of a function F: IR ⟶ IR which satisfies the conditions for F to be a lift of a circle map without assuming the continuity. In particular, F is continuous from the right for the points of Z if its continuity on [0, 1) is assumed. Consequently, F is also continuous on IR–Z.

Lemma 4.1: Assume that F: IR ⟶ IR is such that, for every x ∈ IR, F(x + 1) = F(x) +

1. For every n ∈ Z and every x ∈ IR, F(x + n) = F(x) + n.

Proof. For n = 0, the outcome is true. Assume for the moment that n ∈ IN. On n, we apply induction. According to the specified condition, it is true for n = 1. Assume that it is true for m. Taking x + m in the given condition, F(x + m +1) = F(x + m) +1 = F(x) + m + 1, using the result for m. Therefore, the result holds for m + 1. Thus, the result holds for every n ≥ 1. Now consider that n ≤ 1. Then n = – m for some m ∈ IN. Furthermore, F(x) = F(x – m + m) = F(x – m) + m, utilising the outcome for a positive

n. Follows that, F(x – m) = F(x) – m. Hence, F(x + n) = F(x) + n for every, n ∈ Z.

Lemma 4.2: Considering that, F: IR ⟶ IR and F(x + 1) – F(x) ∈ Z for each x ∈ IR then,

i) F(x + n) – F(x) ∈ Z for every n ∈ IN and x ∈ IR.
ii) F(x + n) – F(x) ∈ Z for every n ∈ Z and x ∈ IR.
iii) Suppose that, F([0, 1)) ⊂ [0, 1), then r(F(x + n)) = F(r(x)) for every n ∈ Z and for each x ∈ IR, particularly, F o r = r o F.
iv) If F o r = r o F, then F([0, 1)) ⊂ [0, 1) and F(x + n) – F(x) ∈ Z for every x

∈ IR and n ∈ Z.

Proof.

i) On n, we apply induction. For n = 1, the outcome remains valid. Assume that for, m, the outcome is true i.e. F(x + m) – F(x) ∈ Z. Currently, after induction, F(x + m + 1) – F(x) = F(x + m + 1) – F(x + m) + F(x + m) – F(x). From the above mentioned condition, we have, F(x + m + 1) – F(x + m) ∈ Z. Using the result for m, F(x + m) – F(x) ∈ Z. Therefore we have, F(x + m + 1) – F(x) ∈ Z.
ii) If n = 0, then the result holds. If n is negative, then n = – m for some m ∈ IN. Then, F(x + n) – F(x) = F(x – m) – F(x) = – (F(x) – F(x – m))
= – (F(x – m + m) – F(x – m)). When we use (i) we have, F(x – m + m) – F(x – m) ∈ Z. Hence, F(x + n) – F(x) ∈ Z.
i) Considering that, x ∈ IR and n ∈ Z. By applying (i), to the integer [x] + n and the number r(x), follows that, F(x + n) – F(r(x)) ∈ Z. Consequently, looking from above, r(F(x + n)) = r(F(r(x))) = F(r(x)) because F(r(x)) ∈ [0, 1).
ii) Let y ∈ [0, 1). Therefore, F(y) = F(r(y)) = r(F(y)). Since r(F(y)) ∈ [0, 1), hence, F(y) ∈ [0, 1). Let x ∈ IR and n ∈ Z. Using above Remark.3.4.(twice) and the given condition, F(x + n) – F(x) ∈ Z.

Remark 4.3: Let x ∈ IR. Then

i) If k ≤ x < k + 1 then k = [x].
ii) If x ∉ Z, then [x] = – [–x] – 1; if x ∈ Z, then [x] = – [– x].
iii) [x + m] = m + [x] for each m ∈ Z.
iv) (x + m) – [x + m] = x – [x] for each m ∈ Z.
v) r(x + m) = r(x) for each m ∈ Z.

Proof. Since, x ∈ IR,

i) Firstly, suppose that k ≠ [x]. If consider, k < [x] then k ≤ [x] – 1. Now, k + 1 ≤ [x] ≤ x < k + 1, a contradiction occurs. Again, if k > [x] then we have [x] ≤ k – 1. Further, [x] + 1 ≤ k ≤ x < [x] + 1, which is again, a contradiction. Hence, k = [x].
ii) Considering, x ∈ Z, we can write, x = [x] and – x = [– x]. Further it follows that, [x] = – [– x]. Again, if we suppose x ∉ Z. Simultaneously, [– x] < – x < [– x] + 1, it follows that, – [–x] – 1 < x < – [–x]. Therefore, by using (i), [x] = – [– x] – 1.
iii) Following that, [x] ≤ x < [x] + 1, we have, m + [x] ≤ m + x < m + [x] + 1. Hence by using (i), [x + m] = m + [x].
iv) With the use of (iii), x + m – [x + m] = x – [x].
v) It follows by using (iv).

Lemma 4.4: Considering that, F: IR ⟶ IR and F(x + 1) – F(x) ∈ Z for each x ∈ IR. Then r(F(x + n)) = r(F(x)) for each x ∈ IR and n ∈ Z.

Proof. By using Lemma 4.2 (ii), we can write that F(x + n) – F(x) ∈ Z. Therefore, by Remark.4.3. (v), r(F(x + n)) = r(F(x)).

Remark 4.5: The converse of the Lemma.4.4, also holds. Let F: IR ⟶ IR be such that r(F(x + 1)) = r(F(x)) for each x ∈ IR, then

i) F(x + 1) – F(x) ∈ Z for each x ∈ IR.
ii) F(x + n) – F(x) ∈ Z for each n ∈ Z and x ∈ IR.

Proof. (i) Following, F(x + 1) = [F(x + 1)] + r(F(x + 1)) and F(x) = [F(x)] + r(F(x)).

Thus, F(x + 1) – F(x) = [F(x + 1)] – [F(x)] since, r(F(x + 1)) = r(F(x)) for each x ∈ IR.

Because, [F(x + 1)] – [F(x)] ∈ Z. Hence, F(x + 1) – F(x) ∈ Z.

(ii) It follows by (i) and Lemma 4.2 (ii).

Lemma 4.6: Let F: IR ⟶ IR be such that F(x + 1) – F(x) ∈ Z for every x ∈ IR. If F([0, 1)) ⊂ [0, 1), then r(F(x + n)) = F(r(x)) for every x ∈ IR and n ∈ Z.

Proof. Let x ∈ IR and n ∈ Z. Since we have, x = [x] + r(x). Therefore, r(F(x + n)) = r(F([x] + n + r(x))). Now by applying Remark 4.3 (iii) to the integer [x]
+ n and the number r(x), we have, r(F([x] + n + r(x)) = r(F(r(x))). Because, r(x) ∈ [0, 1) in addition to the stated condition, F(r(x)) ∈ [0, 1). Therfore, r(F(r(x))) = F(r(x)). Hence, F(r(x)) = r(F([x] + n + r(x))) = r(F(x + n)).

Remark 4.7: Let F: IR ⟶ IR be such that F(x + 1) – F(x) ∈ Z for every x ∈ IR. F([0, 1)) ⊂ [0, 1) if and only if F(r(x)) = r(F(x)) for every x ∈ IR.

Proof. Let F([0, 1)) ⊂ [0, 1). Put n = 0 in Lemma.4.6, F(r(x)) = r(F(x)) for x ∈ IR. Conversely, suppose y ∈ [0, 1). Therefore, F(y) = F(r(y)) = r(F(y)).

Since r(F(y)) ∈ [0, 1). Hence, F(y) ∈ [0, 1).

Lemma.4.8. Let Y be a set and H ⊂ IR. Let g: H ⟶ Y, f: Y⟶ Y and F: H ⟶ H. If f o g = g o F, then f^n o g = g o F^n for every n ∈ IN.
Proof. We prove by induction. It is true for n = 1. Let us suppose that it is true for m. Now, f^{m+1} o g = (f o f^m) o g = f o (f^mo g) = f o (g o F^m) = (f o g) o F^m = (g o F) o F^m = g o (F o F^m) = g o F^{m+1}.

5. Composition of circle map with lift map

Let f: S^1⟶ S^1. The function, π^{-1} o f o π is from [0, 1) to [0, 1). We shall denote π^{-1} o f o π by f*.

Remark 5.1: Let f: S^1⟶ S^1 be a circle map and F be a lift for f with F: IR ⟶ IR. Since π o F = f o π, F = π^{-1} o f o π. Since π^{-1} o f o π: [0, 1) ⟶ [0, 1) then F = f* on [0, 1).

Lemma 5.2: Let f: S^1⟶ S^1. Then $(f^*)^n$ = π^{-1} o f^n o π.
Proof. Considering Lemma.4.8, we use Y = S^1, H = [0, 1), F = f* and g = π. Thus, we get π o $(f^*)^n$ = f^n o π. Which implies that π^{-1} o π o $(f^*)^n$ = π^{-1} o f^n o π. Hence, $(f^*)^n$ = π^{-1} o f^n o π.

Lemma 5.3: Let f: S^1⟶ S^1. Let F: IR ⟶ IR. Then f^n o π = π o F^n.
Proof. Take H = IR, Y = S^1 and g = π in Lemma.4.8, we get f^n o π = π o F^n.

Proposition 5.4: Suppose F: IR ⟶IR and f: S^1⟶ S^1. If f o π = π o F, then for each x ∈ IR, there exits k_x ∈ Z such that F(x + 1) – F(x) = k_x.
Proof. By definition of π function, π(x + 1) = π(x). Therefore, f(π(x + 1)) = f(π(x)). Thus, we have π(F(x + 1)) = π(F(x)). Now again by Remark 4.5 (i), F(x + 1) – F(x) ∈ Z. So there exits k_x∈ Z such that F(x + 1) – F(x) = k_x.

Remark 5.5: Consider, f: S^1⟶ S^1 be a circle map. For given x ∈ IR, let us define F(x) = [x] + f*(r(x)). Suppose if, x ∈ [0, 1) then we have, [x] = 0. Thus, F(x) = f*(x) on [0, 1). Considering that, x ∈ IR. Thus, by the use of Remark 3.4 (iii), (π o F)(x) = π(F(x)) = π([x] + f*(r(x)) = π(f*(r(x))). Therefore, π o F = π o f*. But π o f* = f o π. Therefore, π o F = f o π. By using definition of F and Remark 3.4 (iii), F(x + 1) = [x + 1] + f*(r(x + 1)) = [x] +1 + f*(r(x)). Hence, F(x +1) = F(x) + 1. Therefore, F satisfies the conditions to be a lift for f, but F need not be continuous.

6. Conclusion

The above mentioned results are expected to be applied in future research on point orbits and transitivity of function in various dynamical systems. There are various subsets of IR in the aforementioned considerations where the continuous fractional part function r exists. There may be other subset of IR on which r is continuous. It is worth investigating to know other subset of IR on which r is continuous, or to have some information about such subsets. The theory and some results relating to lift maps are then presented, including an analysis of the continuity aspect of the lift map and its composition with circle maps. Additionally, the idea of continuity in connected spaces is explained, and some key findings are presented.

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